Malliavin Calculus

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1 Introduction.

Malliavin calculus was initiated in 1976 with the work [9] by Paul Malliavin and is essentially an infinite dimensional differential calculus on the Wiener space. Its initial goal was to give conditions insuring that the law of a random variable has a density with respect to Lebesgue measure as well as estimates for this density and its derivatives. When the random variables are solutions of stochastic differential equations, these densities are heat kernels and Malliavin used Hörmander type assumptions on the corresponding operators, thus providing a probabilistic proof of a Hörmander type theorem for hypoelliptic operators.

The theory was much developed in the eighties by Stroock, Bismut and Watanabe, among others (we refer to [14] as well as to [10], where a list of references can be found). These last years Malliavin calculus had a big success in probabilistic numerical methods, mainly in the field of stochastic finance ([13]). But the theory has also been applied to other fields of mathematics and physics, notably in statistical mechanics and in statistical hydrodynamics. Also one should remember that Wiener measure can be viewed as an "imaginary time" (but well defined) counterpart of Feynman's "measure" for quantum systems. A stochastic calculus of variations for Wiener functionals could not be irrelevant to the path integral approach to quantum theory.

Another field of application worth to be mentionned is the study of representations of stochastic oscillatory integrals with quadratic phase function and their stationary phase estimation. For this complexification of the Wiener space must be properly defined ([12]).

In order to give a flavour of what Malliavin calculus is all about, let us consider a second order differential operator in $\mathbb{R}^d$ of the form

$$\mathcal{A} = \frac{1}{2}\sum_{i,j=1} a^{ij}\partial^2_{i,j} + \sum_i b^i \partial_i$$

with smooth bounded coefficients and such that the matrix a is symmetric and nonnegative, admitting a square root σ. The corresponding Cauchy value problem consists in finding a smooth solution $u(t,x)$ of

$$\frac{\partial u}{\partial t} = \mathcal{A}u, \qquad u(0,.) = \phi(.) \tag{1.1}$$

Then there exists a transition probability function $p(t,x,.)$ such that

$$u(t,x) = \int_{\mathbb{R}^d} \phi(y)p(t,x,dy)$$

When $p(t,x,dy) = p(t,x,y)dy$, the function p is the heat kernel associated to the operator $\mathcal{A}$ and from equation (1.1) one may deduce Focker-Plank's equation for p.

Since Kolmogorov we know that it is possible to associate to such a second order operator a stochastic family of curves like a deterministic flow is associated with a vector field. This stochastic family is a Markov process, $\xi_x(t)$, which is adapted to the increasing family $\mathcal{P}_\tau,\ \tau \in [0,1]$, of sigma-fields generated by the past events, i.e., $u(\tau) \in \mathcal{P}_\tau$ for every τ.

Itô calculus allows to write the stochastic differential equation (s.d.e.) satisfied by ξ,

$$d\xi(t) = \sigma(\xi_x(t))dW(t) + b(\xi_x(t))dt, \qquad \xi_x(0) = x \tag{1.2}$$

where $W(t)$ stands for $\mathbb{R}^d$-valued Brownian motion (c.f. [369] in this Encyclopedia). Then p is the image of the Wiener measure μ (the law of Brownian motion), namely $p(t,x,.) = \mu \circ \xi_x^{-1}(t)(.)$ and we have the representation

$$u(t,x) = E_\mu(\phi(\xi_x(t)))$$

The following criterium for absolutely continuity of measures in finite dimensions hold:

Lemma. *If γ is a probability measure on $\mathbb{R}^d$ and, for every $f \in C_b^\infty$,*

$$|\int \partial_i f d\gamma| \leq c_i ||f||_\infty$$

where $c_i, i = 1,\ ..., d$, are constants, then γ is absolutely continuous with respect to Lebesgue measure.

Now one can think about Wiener measure as an infinite (actually continuous) product of finite dimensional Gaussian measures. Considering the toy model of the abovementionned situation in one dimension, we replace Wiener measure by $d\gamma(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$ and look at the process at a fixed time as a function g on $\mathbb{R}$. In order to apply the lemma and study the law of g one would write:

$$\int (f' \circ g) d\gamma = \int \frac{(f \circ g)'}{g'} d\gamma$$

and then integrate by parts to obtain $\int(f \circ g)\rho d\gamma$. A simple computation shows that $\rho(x) = \frac{g^{''}+xg^{'}}{(g^{'})^2}$, and also that the non degeneracy of the derivative of g plays a rôle in the existence of the density.

To work with functionals on the Wiener space one needs an infinite dimensional calculus. Of course, other (Gateaux, Fréchet) calculus on infinite dimensional settings are already available but the typical functionals we are dealing with, solutions of s.d.e's, are not continuous with respect to the underlying topology, nor even defined at every point, but only almost everywhere. Malliavin calculus, as a Sobolev differential calculus, requires very little regularity, given that there is no Sobolev imbedding theory in infinite dimensions.

2 Differential calculus on the Wiener space.

We restrict ourselves to the classical Wiener space, although the theory may be developed in abstract Wiener spaces, in the sense of Gross. For a description of this theory as well as of Segal's model developed in the fifties for the needs of Quantum Field Theory we refer to [10].

Let $\mathcal{H}$ be the Cameron-Martin space, $\mathcal{H} = \{h : [0,1] \to \mathbb{R}^d$ s.t. $\dot{h}$ is square integrable and $h(t) = \int_0^t \dot{h}(\tau)d\tau\}$, which is a separable Hilbert space with scalar product $< h_1, h_2 >= \int_0^1 \dot{h}_1(\tau).\dot{h}_2(\tau)d\tau$. The classical Wiener measure will be denoted by μ; it is realized on the Banach space X of continuous paths on the time interval $[0,1]$ starting from zero at time zero, a space where $\mathcal{H}$ is densily imbedded. In finite dimensions Lebesgue measure can be characterized by its invariance under the group of translations. In infinite dimensions there is no Lebesgue measure and this invariance must be replaced by quasi-invariance for translations of Wiener measures (Cameron-Martin admissible shifts). We recall that, if $h \in \mathcal{H}$, Cameron-Martin theorem states that

$$E_\mu(F(\omega + h)) = E_\mu(F(\omega)exp(\int_0^1 \dot{h}(\tau).d\omega(\tau) - \frac{1}{2}\int_0^1 |\dot{h}(\tau)|^2 d\tau))$$

where $d\omega$ denotes Itô integration.

For a cylindrical "test" functional $F(\omega) = f(\omega(\tau_1), \ ..., \omega(\tau_m))$, where $f \in C_b^\infty(\mathbb{R}^m)$ and $0 \leq \tau_1 \leq \ ... \leq \tau_m \leq 1$, the derivative operator is defined by

$$D_\tau F(\omega) = \sum_{k=1}^m \mathbf{1}_{\tau<\tau_k}\partial_k f(\omega(\tau_1), \ ..., \omega(\tau_m)) \tag{2.1}$$

This operator is closed in $W_{2,1}(X;\mathbb{R})$, the completion of the space of cylindrical functionals with respect to the Sobolev norm

$$||F||_{2,1} = E_\mu||F||^2 + E_\mu \int_0^1 |D_\tau F|^2 d\tau$$

Define F to be $\mathcal{H}$ - differentiable at $\omega \in X$ when there exists a linear operator $\nabla F(\omega)$ such that, for all $h \in \mathcal{H}$,

$$F(\omega + h) - F(\omega) =< \nabla F(\omega), h > +o(||h||_H), \quad \text{as} \quad ||h|| \to 0$$

Then D_τ desintegrates the derivative in the sense that

$$D_h F(\omega) \equiv < \nabla F(\omega), h >= \int_0^1 D_\tau F(\omega).\dot{h}(\tau) d\tau \tag{2.2}$$

Higher (r) order derivatives, as r-linear functionals, can be considered as well in suitable Sobolev spaces.

Denote by δ the L^2_μ adjoint of the operator ∇, i.e., for a process $u : X \to \mathcal{H}$ in the domain of δ, the divergence $\delta(u)$ is characterized by

$$E_\mu(F\delta(u)) = E_\mu(\int_0^1 D_\tau F.\dot{u}(\tau) d\tau) \tag{2.3}$$

For an elementary process u of the form $u(\tau) = \sum_j F_j(\tau \wedge \tau_j)$ where the F_j are smooth random variables and the sum is finite, the divergence is equal to

$$\delta(u) = \sum_j F_j \omega(\tau_j) - \sum_j \int_0^{\tau_j} D_\tau F_j d\tau$$

The characterization of the domain of δ is delicate, since both terms in this last expression are not independently closable. It can be shown that $W_{1,2}(X; H)$ is in the domain of δ and that the following "energy" identity holds:

$$E_\mu(\delta(u))^2 = E_\mu ||u||^2_H + E_\mu \int_0^1 \int_0^1 D_\tau \dot{u}_\sigma . D_\sigma \dot{u}_\tau d\sigma d\tau$$

Notice that when u is adapted to $\mathcal{P}_\tau$ Cameron-Martin-Girsanov theorem implies that the divergence coincides with Itô stochastic integral $\int_0^1 \dot{u}(\tau).d\omega(\tau)$ and, in this adapted case, the last term of the energy identity vanishes. We recover the well known Itô isometry which is at the foundation of the construction of this integral. When the process is not adapted the divergence turns out to coincide with a generalization of Itô integral, first defined by Skorohod.

The relation (2.3) is an integration by parts formula with respect to the Wiener measure μ, one of the basic ingredients of Malliavin calculus. This formula is easily generalized when the base measure is absolutely continuous with respect to μ.

Considering all functionals of the form $\mathcal{P}(\omega) = Q(\omega(\tau_1), \ ..., \omega(\tau_m))$ with Q a polynomial on $\mathbb{R}^d$, the Wiener chaos of order n, $\mathcal{C}_n$, is defined as $\mathcal{C}_n = \mathcal{P}_n \bigotimes \mathcal{P}^\perp_{n-1}$, where $\mathcal{P}_n$ denote the polynomials on X of degree $\leq n$. The Wiener-chaos decomposition $L^2_\mu(X) = \bigoplus_{n=0}^\infty \mathcal{C}_n$ holds. Denoting by Π_n the orthogonal projection onto the chaos or order n we have

$$< \nabla(\Pi_{n+1} F), h >= \Pi_n(< \nabla F, h >)$$

The derivative D_u corresponds to the annihilation operator $A(u)$ and the divergence $\delta(u)$ to the creation operator $A^+(u)$ on bosonic Fock spaces.

An important result, known as the Clark-Bismut-Ocone formula, states that any functional $F \in W_{1,2}(X; \mathbb{R})$ can be represented as

$$F = E_\mu(F) + \int_0^1 E_\tau(D_\tau F) d\omega(\tau)$$

where E_τ denotes the conditional expectation with respect to the events prior to time τ (or, for short, the past $\mathcal{P}_\tau$ of τ).

The Ornstein-Ulhenbeck generator (or minus number operator) is defined by $\mathcal{L}F = -\delta\nabla F$. On cylindrical functionals $F(\omega) = f(\omega(\tau_1), \ ..., \omega(\tau_m))$ it has the form:

$$\mathcal{L}F(\omega) = \sum_{i,j} \sigma_i \wedge \sigma_j \partial^2_{i,j} f(\omega(\tau_1), \ ..., \omega(\tau_m)) - \sum_j \omega(\tau_j)\partial_j f(\omega(\tau_1), \ ..., \omega(\tau_m))$$

where i, j denote multi (d)-dimensional indexes.

As a multiplicative operator on the Wiener chaos decomposition $\mathcal{L}F = -\sum_n n\Pi_n F$. It is the generator of a positive μ - selfadjoint semigroup, the Ornstein-Ulhenbeck semigroup, formally given by $T_t F = \sum_n e^{-nt}\Pi_n F$. Another familiar representation of this semigroup is Mehler formula,

$$T_t F(\omega) = E_\mu(F(e^{-t}\omega + \sqrt{(1 - e^{-2t}}\upsilon) d\mu(\upsilon))$$

Considering the map $X \to \mathbb{R}^m$, $\omega \to (\omega(\tau_1), \ ..., \omega(\tau_m))$, the image of this operator is the Ornstein-Ulhenbeck generator (corresponding to the Langevin equation) on $\mathbb{R}^m$ with Euclidean metric defined by the matrix $\tau_i \wedge \tau_j$.

The fundamental theorem concerning existence of the density laws of Wiener functionals is the following:

Theorem. *Let F be a $\mathbb{R}^d$-valued Wiener functional such that F^i and $\mathcal{L}F^i$ belong to L^4_μ for every $i = 1, \ ..., d$. If the covariance matrix*

$$< \nabla F^i, \nabla F^j >_{\mathcal{H}}$$

is almost surely invertible, then the law of F is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$.

Under more regularity assumptions smoothness of the density is also derived. On the other hand, the integrability assumptions on $\mathcal{L}$ can be replaced by integrability of the second derivatives, due to Krée-Meyer inequalities on the Wiener space.

We remark that, although equivalent, the initial formulation ([9]) of Malliavin calculus was different, relying on the construction of the two-parameter process associated to $\mathcal{L}$ and on its properties. In the early eighties the theory was elaborated, the main applications being the study of of heat kernels (c.f. for example [15], [7] and [2]). Starting from a s.d.e.(1.2) it is possible to apply these techniques to obtain existence and smoothness of the transition probability function $p(t,x,y)$ if the vector fields $Z_i = \sum_j \sigma^{ij} \frac{\partial}{\partial x_j}$ together with their Lie brackets generate the tangent space for "sufficientely many" (in terms of probability) paths. These results shed a new light on Hörmander theorem for partial differential equations.

3 Quasi-sure analysis.

Quasi-sure analysis is a refinement of classical probability theory and, generally speaking, replaces the fact, due to Sobolev imbedding theorems, functions in finite dimensions belonging to Sobolev classes are in fact smooth. We work in classical probability up to sets of probability zero; in quasi-sure analysis negligible sets are smaller and are those of capacity zero. This is the class of sets which are not charged by any measure of finite energy.

Under a non-degenerate map Wiener measure and more general Gaussian measures may be desintegrated through a co-area formula. This principle, developed by Malliavin and co-authors (c.f. [10] and references therein) implies that a property which is true quasi-surely will also hold true almost surely under conditionning by such a map. One can use this principle to study finer properties of stochastic differential equations. It was also used in [11] to transfer properties from path to loop groups. A pinned Brownian motion, for example, is well defined in quasi-sure analysis. It is possible to treat anticipative problems using quasi-sure analysis by solving the adapted problem after restriction of the solution to the finite-codimensional manifold which describes the anticipativity. These methods have been also applied to the computation of Lyapunov exponents of stochastic dynamical systems ([8]). With a geometry of finite co-dimensional manifolds of Wiener spaces well established it is reasonable to think about applications to cases where such submanifols correspond to level surfaces of invariant quantities for infinite dimensional dynamical systems (c.f. [3] for an example of such a situation).

The (p,r)-capacity of an open subset O of the Wiener space is defined by

$$cap_{p,r}(O) = inf\{||\phi||^p_{W_{2r}},\ \phi \geq 0,\ \phi \geq 1\ \ \mu\text{-a.s. on } O\}$$

and, for a general set B, $cap_{p,r}(B) = inf\{cap_{p,r}(O) : B \subset O, O\ open\}$. A set is said to be slim if all its (p,r)-capacities are zero. For $\Phi \in W_\infty$, the space of functionals with every Malliavin derivative belonging to all L^p_μ, there exists a redifinition of Φ, denoted by Φ^*, which is smooth and defined on the complement of a slim set.

Following [1], let $G \in W_\infty(X;\mathbb{R}^d)$ be of maximal rank and non degenerate in the sense that the inverse of

$$(det\ \Phi)^2(\omega) = det\ (< \nabla\Phi^i(\omega), \nabla\Phi^j(\omega) >)$$

belongs to W_∞. Then for every functional $G \in W_\infty$ the measures $\mu \circ \Phi^{-1}$ and

$(G\mu)\circ\Phi^{-1}$ are absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$ and have C^∞ Radon-Nikodym derivatives. If

$$\rho(\lambda) = \frac{d\mu\circ\Phi^{-1}}{d\lambda} \quad \text{and} \quad \rho_G(\lambda) = \frac{d(G\mu)\circ\Phi^{-1}}{d\lambda}$$

the function $\lambda \to \frac{\rho_G(\lambda)}{\rho(\lambda)}$ will be smooth in the open set $\mathcal{O} = \{\lambda : \rho(\lambda) > 0\}$.

For every $\lambda \in \mathcal{O}$ it is possible to define (up to slim sets) a submanifold of the Wiener space of co-dimension d, $\mathcal{S}_\lambda = (\Phi^*)^{-1}(\lambda)$, as well as a measure μ_S satisfying:

$$\int_{\mathcal{S}_\lambda} G^* d\mu_S(\omega) = E^{\Phi(\omega)=\lambda}(G) = \frac{\rho_G(\lambda)}{\rho(\lambda)}$$

for every $G \in W_\infty$. This measure does not charge slim sets.

The area measure $\aleph$ on the submanifold $\mathcal{S}_\lambda$ is defined by

$$\int F^* d\aleph = \rho(\lambda)\int F^*(\omega) \;\; det\ (<\nabla\Phi^i(\omega), \nabla\Phi^j(\omega)>)^{\frac{1}{2}} d\mu_S(\omega)$$

The following co-area formula on the Wiener space

$$\int_X f(\Phi(\omega))F(\omega)(det\ \Phi)(\omega)d\mu(\omega) = \int_{\mathbb{R}^d} f(\lambda)\int_{\mathcal{S}_\lambda} F^*(\omega)d\aleph(\omega)d\lambda$$

was proved in [1].

4 Calculus of variations in a non Euclidean setting.

Let M be a d-dimensional compact Riemannian manifold with metric $ds^2 = \sum_{i,j} g_{i,j}dm^i dm^j$. The Laplace-Beltrami operator is expressed in the local chart by

$$\Delta_M = g^{i,j}\frac{\partial^2 f}{\partial m^i \partial m^j} - g^{i,j}\Gamma^k_{i,j}\frac{\partial f}{\partial m^k}$$

where $\Gamma^k_{i,j}$ are the Christoffel symbols associated with the Levi-Civita connection. The corresponding Brownian motion p_w is locally expressed as a solution of the s.d.e.

$$dp^i(t) = a^{i,j}(p(t))dW_j(t) - \frac{1}{2}g^{j,k}\Gamma^i_{j,k}(p(t))dt$$

with $p(0) = m_0 \in M$ and where $a = \sqrt{g}$. Its law on the space of paths $\mathbb{P}(M) = \{p : [0,1] \to M,\ p \text{ continuous},\ p(0) = m_0\}$ will be denoted by ν.

How can we develop differential calculus and geometry on the space $\mathbb{P}(M)$? An infinite dimensional local chart approach is dellicate, due to the difficulty of finding an atlas in which the changes of charts preserve the measures. A possibility, developed in [4], consists in replacing the local chart approach by the Cartan-like methodology of moving frames. The canonical moving frame in this framework is provided by Itô stochastic parallel transport. Nevertheless a new difficulty arises: the parallel transport will not be differentiable in the Cameron-Martin sense described in last paragraph.

Recall that a frame above m is an Euclidean isometry $r : \mathbb{R}^d \to T_m(M)$ onto the tangent space. $O(M)$ denotes the collection of all frames above M and $\pi(r) = m$ the canonical projection. $O(M)$ can be viewed as a parallelized manifold for there exist canonical differential forms (θ, ω) realizing for every r an isomorphism between $T_r(O(M))$ and $\mathbb{R}^d \times so(d)$.

If $A_\alpha, \alpha = 1, ..., d$, denote the horizontal vector fields, which are defined by $< \theta, A_\alpha >= \varepsilon_\alpha$, $< \omega, A_\alpha >= 0$, where ε_α are the vectors of the canonical basis of $\mathbb{R}^d$, then the horizontal Laplacian in $O(M)$ is the operator

$$\Delta_{O(M)} = \sum_{\alpha=1}^{d} A_\alpha^2$$

and we have $\Delta_{O(M)}(f o\pi) = (\Delta_M f)o\pi$. The Laplacians on M and on $O(M)$ inducing two probability measures, the canonical projection realizes an isomorphism between the corresponding probability spaces.

The Stratonovich stochastic differential equation

$$dr_\omega = \sum_\alpha A_\alpha(r_\omega) o d\omega^\alpha, \qquad r_\omega(0) = r_0$$

with $\pi(r_0) = m_0$ defines the lifting to $O(M)$ of the Itô parallel transport along the Brownian curve and we write $t^p_{\tau\leftarrow 0} r_0 = r_\omega(\tau)$. Itô map was defined by Malliavin as the map $I : X \to \mathbb{P}(M)$ given by

$$I(\omega)(\tau) = \pi(r_\omega(\tau))$$

This map is a.s. bijective and we have $\nu = \mu \circ I^{-1}$; therefore it provides an isomorphism of measures from the curved path space to the "flat" Wiener space.

For a cylindrical functional $F = f(p(\tau_1), ..., p(\tau_m))$ on $\mathbb{P}(M)$ the derivatives are defined by

$$D_{\tau,\alpha} F(p) = \sum_{k=1}^{m} \mathbf{1}_{\tau<\tau_k} (t^p_{0\leftarrow\tau_k}(\partial_k F)|\varepsilon_\alpha)$$

The derivative operator is closable in a suitable Sobolev space.

It would be reasonable to think that the differentiable structure considered in the Wiener space would be conserved through the isomorphism I and that the tangent space of $\mathbb{P}(M)$ would consist of transported vectors from the tangent space to X, namely Cameron-Martin vectors. Let us take a map $Z_p(\tau) \in T_{p(\tau)}(M)$ such that $z(\tau) = t^p_{0\leftarrow\tau} Z_p(\tau)$ belongs to the Cameron-Martin space $\mathcal{H}$.

In order to transfer derivatives to the Wiener space we need to differentiate the Itô map. We have ([4]):

Theorem. *The Jacobian matrix of the flow $r_0 \to r_\omega(\tau)$ is given by the linear map $J_{\omega,\tau} = (J^1_{\omega,\tau}, J^2_{\omega,\tau}) \in GL(\mathbb{R}^d \times so(d))$ defined by the system of Stratonovich s.d.e's*

$$\begin{cases} d_\tau J^1_{\omega,\tau} = \sum_{\alpha=1}^d (J^1_{\omega,\tau})_\alpha o d\omega_\alpha(\tau) \\ d_\tau J^2_{\omega,\tau} = \sum_{\alpha=1}^d \Omega(J^1_{\omega,\tau}, \varepsilon_\alpha) o d\omega_\alpha(\tau) \end{cases}$$

where Ω denotes the curvature tensor of the underlying manifold read on the frame bundle.

From this result we can deduce the behavior of the derivatives transfered to the Wiener space, a result whose origin is due to B. Driver. We have, for a "vector field" $Z_p(\tau)$ on $\mathbb{P}(M)$ as above,

$$(D_Z F) o I = D_\xi(F o I)$$

with ξ solving

$$\begin{cases} d\xi(\tau) = \dot{z}(\tau) d\tau + \rho o d\omega(\tau) \\ d\rho(\tau) = \Omega(o d\omega(\tau), z(\tau)) \end{cases}$$

The process ξ is no longer Cameron-Martin space valued. Nevertheless it satisfies an s.d.e. with an antisymmetric diffusion coefficient (given by the curvature) and therefore, by Levy's theorem, it still corresponds to a transformation of the Wiener space that leaves the measure quasi-invariant. We extend, accordingly, the notion of tangent space in the Wiener space to include processes of the form $d\xi^\alpha = a^\alpha_\beta d\omega^\beta + c^\alpha d\tau$, with $a^\alpha_\beta + a^\beta_\alpha = 0$. These were called *tangent processes* in [4].

Another important consequence of last theorem is the integration by parts formula in the curved setting, initially proved by Bismut ([2]):

$$E_\nu(D_Z F) = E_\mu((F o I) \int_0^1 [\dot{z} + \frac{1}{2} Ricci(z)] d\omega(\tau))$$

where *Ricci* is the Ricci tensor of M read on the frame bundle.

5 Some applications.

We already mentionned that Malliavin calculus has been applied to various domains connected with physics. We shall describe here some of its relations with elementary quantum mechanics.

Feynman gave a path space formulation of quantum theory whose fundamental tool is the concept of transition element of a functional $F(\omega)$ between any two L^2-states ψ_s and ϕ_u, for paths ω defined on a time interval $[s, u]$:

$$< F >_S \equiv < \phi | F | \psi >_S = \int\int\int_\Omega \psi_s(x) e^{\frac{i}{\hbar} S_L(\omega, u-s)} F(\omega) \bar{\phi}_u(z) \mathcal{D}\omega dx dz \qquad (4.1)$$

This is a shorthand for the time discretization version along broken paths ω interpolating linearly between point $x_j = \omega(t_j)$, $t_j = j\frac{u-s}{N}$, $j = 0, 1, \ ..., N$. In (4.1) $\hbar$ is Planck's constant and $S = S_L$ denotes the action functional with Lagrangian L of the underlying classical system. For a particle with mass m in a a scalar potential V on the real line,

$$S_L(\omega, u - s) = \int_s^u (\frac{m}{2}\dot{\omega}^2(\tau) - V(\omega(\tau))d\tau \qquad (4.2)$$

The "$\mathcal{D}\omega$" of (4.1) is used as a Lebesgue measure, although there is no such thing in infinite dimensions. More generally, the construction of measures or integrals on the various path spaces required for general quantum systems is still nowadays a field of investigation.

When $F = 1$ and $\bar{\phi}_u$ (the complex conjugate of ϕ_u) reduces to a Dirac mass at z, (4.1) is the path integral representation of the solution $\psi(x, u)$ of the initial value problem in L^2:

$$\begin{cases} i\hbar\frac{\partial\psi}{\partial u} = H\psi \\ \psi(x, s) = \psi_s(x) \end{cases} \qquad (4.3)$$

where $H = -\frac{\hbar^2}{2}\Delta + V$ and when S_L is as in (4.2). Feynman's framework is time-symmetric on I: when $\psi_s = \delta_x$ (still for $F = 1$), (4.1) provides a path integral representation of the solution of the final value problem for $\bar{\phi}(z, s)$.

According to Feynman *"it would be possible to use the integration by parts formula*

$$< \frac{\delta F}{\delta\omega(s)} >= -\frac{i}{\hbar} < F\frac{\delta S}{\delta\omega(s)} > \qquad (4.4)$$

as a starting point to define the laws of quantum mechanics" ([6], p.173). The functional derivative corresponds to variations of the underlying paths in directions $\delta\omega$ and $\delta F = \int \frac{\delta F}{\delta\omega(s)}\delta\omega(s)ds$ to a L^2 analog of (2.2).

Its first consequence, when $F = 1$, is the path space counterpart of Newton's law, in the elementary case (4.2),

$$< m\ddot{\omega} >_{S_L} = - < \nabla V(\omega) >_{S_L} \qquad (4.5)$$

where the left-hand side involves a time discretization of the second derivative. When $F(\omega) = \omega(t)$, Feynman obtains the path space version of Heisenberg commutation relation between position and momentum observables:

$$< \omega(t)\frac{\omega(t) - \omega(t-\epsilon)}{\epsilon} >_{S_L} - < \frac{\omega(t+\epsilon) - \omega(t)}{\epsilon}\omega(t) >= i\frac{\hbar}{m} \qquad (4.6)$$

and from this the crucial fact that " *Quantum mechanical paths are very irregular. However, these irregularities average out over a reasonable lenght of time to produce a reasonable drift or average velocity*"([6], pg. 177).

A probabilistic interpretation (c.f. [5]) of Feynman's calculus uses (Bernstein) diffusion processes solving the s.d.e.

$$dz(t) = (\frac{\hbar}{m})^{\frac{1}{2}} dW(t) + \frac{\hbar}{m}\nabla log\ \eta(z(t),t)dt \tag{4.7}$$

where the drift stems from a positive solution of the Euclidean version of the above final value problem for $\bar{\phi}$,

$$\begin{cases} \hbar\frac{\partial\eta}{\partial t} = H\eta \\ \eta(x,u) = \eta_u(x) \end{cases} \tag{4.8}$$

For any regular function f we can make sense of the "continuous limit"

$$\mathbb{D}f(z(t),t) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}E_t[f(z(t+\epsilon),t+\epsilon) - f(z(t),t))] \tag{4.9}$$

where E_t denotes conditional expectation with respect to the past $\mathcal{P}_t$ and check, indeed, that $\mathbb{D}z(t) = \frac{\hbar}{m}\nabla log\ \eta(z(t),t)$ is Feynman's "reasonable drift". Using Feynman-Kac formula, one shows that the diffusions (4.7) have laws which are absolutely continuous with respect to the Wiener measure of parameter $\frac{\hbar}{m}$, with Radon-Nikodym density given by

$$\rho(z) = \frac{\eta(z(u),u)}{\eta(z(s),s)}exp(-\frac{1}{\hbar}\int_0^1 V(z(\tau))d\tau)$$

We can, therefore, use Malliavin calculus on the path space of these diffusions and the associated integration by parts formula, to make sense of (4.4) and all its consequences.

The probabilistic counterpart of the time symmetry of Feynman's framework is interesting: Heisenberg's original argument to deny the existence of quantum trajectories (1927) was that any position can be associated with two velocities. Feynman's interpretation (4.6) and the definition (4.9) suggest that this has to do with a past or future conditionning at time t. Indeed there is another description of diffusions $z(t)$ with respect to a family of future σ-fields, using the Euclidean version of the initial value problem for ψ, underlying (4.1). Another drift, built on the model of the one of (4.7), results, and Feynman's commutation relation (4.6) becomes rigorous (without, of course, the factor i).

We refer to [5] for a development of this approach using Malliavin calculus.

See also:

Euclidean field theories. Functional integration in quantum physics. Path integrals. Stochastic differential equations.

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