# RIEMANNIAN GEOMETRY ON THE PATH SPACE

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## 1. Introduction

This paper is a survey of articles published by the authors in the last six years on the subject together with related work in the literature.

When trying to construct a Riemannian geometry on the path space of a Riemannian manifold several approaches could be thought about. The local chart approach, considering the path space as an infinite dimensional manifold and the basic tangent space the Cameron-Martin Hilbert space, leads to the study of the so-called Wiener-Riemann manifolds [18]. Several difficulties appear in this study, namely the difficulty of finding an atlas such that the change of charts is compatible with the probabilistic structure (preserves the class of Wiener measures together with the Cameron-Martin type tangent spaces) and the non-availability of an effective computational procedure in the local coordinate system. Indeed, in infinite

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dimensions, the summation operators of differential geometry become very often divergent series.

Another approach to construct a geometry could be the use of a frame bundle. The corresponding object to the bundle of orthonormal frames would be the group of unitary transformations of a Hilbert space. Without further restrictions, this group seems too large to be considered in an efficient way.

But the path space is more than a space endowed with a probability: time and the corresponding Itô filtration provide a much richer structure. In particular, the parallel transport over Brownian paths can be naturally defined by a limiting procedure from ODEs to SDEs. The stochastic parallel transport defines a canonical moving frame on the path space: the point of view we have adopted is the one of replacing systematically the machinery of local charts by the method of moving frames (as in Cartan theory [3]). In this way it is possible to transfer geometrical quantities of the path space to the classical Wiener space and use Itô calculus to renormalize the apriori divergent expressions. An effective computational procedure is then achieved, where Stochastic Analysis and Geometry interact, not only on a technical level, but in a deeper way: Stochastic Analysis makes it possible to define geometrical quantities, Geometry implies new results in Stochastic Analysis.

1.1. **Some geometrical preliminaries.** Let $M$ be a Riemannian manifold of dimension $d$, that we shall always assume to be compact. $O(M)$ denotes the bundle of orthonormal frames over $M$, namely

$$O(M) = \big\{(m,r): \; r:\mathbb{R}^d \to T_m(M) \text{ is a Euclidean isometry, } m \in M\big\}$$

and $\pi : O(M) \to M$, $\pi(r) = m$ the canonical projection.

A smooth section of $O(M)$, namely a smooth map $\sigma : M \to O(M)$ such that $\pi \circ \sigma = \mathrm{Id.}$ is called a *Riemannian parallelism.* In Cartan's theory of moving frame Geometry, an *orthonormal moving frame* is the data of $d$ unitary vector fields $B_k$ on $M$. Denote by $\Theta_k$ the corresponding dual differential forms, $\langle z, \Theta_k\rangle = (z \mid B_k)$. Then the structural equations are defined as

$$d\Theta_k = a_k^{i\,j}\,\Theta_i \wedge \Theta_j,$$

where $a_k^{i\,j}$ are (uniquely defined) functions on $M$.

The brackets of the vector fields $B_k$ are then expressed by

$$[B_k, B_l] = -\sum a_i^{kl} B_i.$$

The *Christoffel differential form* associated to $\Theta$ is the so($d$) 1-differential form $\Gamma$ such that, for all vector fields $A$ and $B$ on $M$ we have

$$\langle A \wedge B, d\Theta\rangle = \Gamma(B)\Theta(A) - \Gamma(A)\Theta(B).$$

Such form exists and is unique. Writing $\Gamma = \Gamma_{ij}^k \Theta^i$, and using the structural equations, we have $a_k^{i\,j} = \Gamma_{ij}^k - \Gamma_{ji}^k$ and the coordinates of $\Gamma$ are uniquely determined by

$$\Gamma_{ij}^k = -\frac{1}{2}[a_k^{i\,j} + a_j^{k\,i} - a_i^{j\,k}].$$

Given a moving frame, the *Levi-Civita covariant derivative* of a vector field $z$ is expressed in the moving frame by

$$\Theta(\nabla_A z) = \mathcal{L}_A \Theta(z) + T(A)\Theta(z),$$

where $\mathcal{L}$ denotes the usual derivative.

It is possible to define on $O(M)$ a structure of parallelized manifold.

Let $\gamma_i$ denote the (unique) geodesic on $M$ such that $\gamma_i(0) = m$, $\frac{d}{dt}\big|_{t=0}\gamma_i(t) = r(e_i)$, where $e_i$, $i = 1, \dots, d$, are the vectors of the canonical basis of $\mathbb{R}^d$, and let $(\gamma_i(t), r_i(t))$ represent the parallel transport of $r$ along $\gamma_i$, defined by the equation

$$\frac{dr_i}{dt} = -\Gamma_{\dot\gamma_i} r_i, \quad r_i(0) = \mathrm{Id}.$$

Then

$$A_i(r) = \frac{d}{dt}\bigg|_{t=0} r_i(t)$$

are the so-called horizontal vector fields on $M$.

Denote by $\Theta$ the form defined by $\langle \Theta, A_i \rangle = (e_i, 0)$. It is a one-form defined on $O(M)$ with values in $\mathbb{R}^d \times \mathrm{so}(d)$, $\Theta = (\theta, \omega)$, where $\omega(m,r) = r^{-1}\, dr$ is the Maurer-Cartan form of the orthogonal group $O(d)$.

The structure equations of the parallelism are given by

$$\begin{cases} d\theta = \omega \wedge \theta, \\ d\omega = \omega \wedge \omega + \Omega(\theta \wedge \theta), \end{cases}$$

where $\Omega$ denotes the curvature tensor:

$$\Omega(A, B, X) = \big(\nabla_A \nabla_B - \nabla_B \nabla_A - \nabla_{[A,B]}\big)X.$$

We define the Laplacian on $O(M)$ by

$$\Delta_{O(M)} = \frac{1}{2}\sum_{k=1}^{d} \mathcal{L}^2_{A_k}.$$

Then for every smooth function on $M$ we have

$$\Delta_{O(M)}(f \circ \pi) = (\Delta_M f) \circ \pi,$$

where $\Delta_M$ denotes the Laplace-Beltrami operator on $M$.

An analogue construction can be performed with respect to any Riemannian connection with torsion. In this case the structure equations are

$$\begin{cases} d\theta = \omega \wedge \theta + T(\theta \wedge \theta), \\ d\omega = \omega \wedge \omega + \Omega(\theta \wedge \theta). \end{cases}$$

If the torsion satisfies the so-called "Driver condition", namely

$$(T(A,B), C) = -(T(C,B), A),$$

then the construction gives rise to the same Laplacian ([10] pg. 347).

1.2. **Stochastic analysis on the Wiener space.** We shall denote by $X$ the classical Wiener space of continuous paths on $\mathbb{R}^d$,

$$X = \big\{x : [0,1] \to \mathbb{R}^d : x \text{ continuous}, x(0) = 0\big\}$$

endowed with the Wiener measure $\mu_0$ and the usual Itô filtration $\mathcal{P}_t$ of the events before time $t$.

A fundamental equality in Stochastic Analysis, that is at the basis of the definition of Itô integral itself is the following energy identity

$$E\left|\int_0^1 u_\tau \cdot dx(\tau)\right|^2 = E\int_0^1 |u_\tau|^2 \, d\tau$$

for $\mathcal{P}_t$-adapted $L^2$ functionals of the Wiener space, and where

$$\int_0^1 u_\tau \cdot dx(\tau) = \int_0^1 u_\tau^\alpha \, dx_\alpha(\tau),$$

using Einstein convention for the sum of indices.

If $F \in L^p(\mu)$ and $z$ is such that $\int_0^1 |\dot{z}_\tau|^2 \, d\tau < +\infty$ ($z$ belongs to the Cameron-Martin space $H^1$), we define

$$D_z F(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\big(F(x + \varepsilon z) - F(x)\big),$$

the limit being taken in the $\mu_0$-a.e. sense. Cameron-Martin-Girsanov theorem implies that

$$E_{\mu_0}(D_z F(x)) = E_{\mu_0}\left(F(x) \int_0^1 \dot{z} \, dx\right), \tag{1.1}$$

that is, Itô integral can be regarded as the dual of a derivation operator on the Wiener space.

For a cylindrical functional $F(x) = f(x(\tau_1), \ldots, x(\tau_m))$, $f$ smooth, let

$$D_\tau F(x) = \sum_{k=1}^m \mathbf{1}_{\tau < \tau_k} \partial_k f(x(\tau_1), \ldots, x(\tau_m)).$$

The operator $D$ is a closed operator on the space $W_{1,2}$, the completion of cylindrical functionals with respect to the norm

$$\|F\|_{1,2}^2 = E_{\mu_0} |F|^2 + E \int_0^1 \|D_\tau F\|^2 \, d\tau,$$

and we can write

$$D_z F = \int_0^1 D_\tau F \cdot \dot{z}_\tau \, d\tau. \tag{1.2}$$

Notice that, if we consider the basic "vector fields" in the Wiener space, $e_{\tau,\alpha}(\sigma) = \mathbf{1}_{\tau < \sigma} \varepsilon_\alpha$, then

$$D_{\tau,\alpha} F = D_{e_{\tau,\alpha}} F.$$

The dual of the derivative, for non adapted processes $z$, is well defined when

$$E \int_0^1 |\dot{z}_\tau|^2 \, d\tau + E \int_0^1 \int_0^1 |D_\sigma \dot{z}(\tau)|^2 \, d\sigma \, d\tau < +\infty.$$

It was discovered by Gaveau and Trauber [15] that the divergence coincides with the Skorohod integral [24], previously defined for non-adapted processes. Following the Nualart-Pardoux-Zakai theory of non-adapted stochastic calculus [22], this integral, that we still denote by $\int_0^1 u \, dx$, can be defined as the limit of the sums

$$\sum_k \mathcal{M}_k(u) \cdot (x(\tau_{k+1}) - x(\tau_k)) - \sum_k \frac{1}{\tau_{k+1} - \tau_k} \int_{\tau_k}^{\tau_{k+1}} \int_{\tau_k}^{\tau_{k+1}} D_\tau u_\sigma \, d\tau \, d\sigma, \tag{1.3}$$

where

$$\mathcal{M}_k(u) = \frac{1}{\tau_{k+1} - \tau_k} \int_{\tau_k}^{\tau_{k+1}} u_\sigma \, d\sigma$$

and is an extension of the Itô integral.

So we have, extending (1.1) to the anticipative case,

$$E_{\mu_0}(D_z F(x)) = E_{\mu_0}\left(F(x) \int_0^1 \dot{z} \cdot dx\right).$$

This implies, in particular,

$$E_{\mu_0}\left(\int_0^1 u\,dx\right)^2 = E\Big(\int_0^1 D_\tau\Big(\int_0^1 u\cdot dx\Big)\cdot u(\tau)\,d\tau\Big)$$

and a commutation relation, namely

$$D_\tau \int_0^1 u\cdot dx = \int_0^1 D_\tau u(\sigma)\cdot dx(\sigma) + u(\tau) \tag{1.4}$$

allows us to derive the corresponding energy identity, which is

$$E\left(\int_0^1 u_\tau\,dx(\tau)\right)^2 = E\int_0^1 |u_\tau|^2\,d\tau + E\int_0^1\int_0^1 D_\tau u_\sigma\cdot D_\sigma u_\tau\,d\tau\,d\sigma. \tag{1.5}$$

Notice that (1.4) reduces to the energy identity for the Itô integral when $u$ is adapted, since the last term vanishes.

We recall here the notion of Stratonovich-Skorohod integral, again following [22]: this integral, that we denote by $\int_0^1 u\circ dx$, is defined as the limit of the sums

$$\sum_k \mathcal{M}_k(u)\cdot(x(\tau_{k+1})-x(\tau_k)). \tag{1.6}$$

Conditions for the existence of such limit are more restrictive than those required for the definition of the Skorohod integral: in particular, some uniform continuity near the diagonal of $[0,1]^2$ is required ([22]). When both integrals exist they are related by

$$\int_0^1 u_\tau\,dx(\tau) = \int_0^1 u_\tau\circ dx(\tau) - \frac{1}{2}\int_0^1 (D_\tau^+\cdot u_\tau + D_\tau^-\cdot u_\tau)\,d\tau, \tag{1.7}$$

where

$$D_\tau^+\cdot u_\tau = \lim_{\sigma\to\tau^+} D_\tau\cdot u_\sigma,$$
$$D_\tau^-\cdot u_\tau = \lim_{\sigma\to\tau^-} D_\tau\cdot u_\sigma.$$

In the case where $u$ is $\mathcal{P}_t$-adapted, $D_\tau^+ u_\tau = 0$ and $\frac{1}{2}\int_0^1 D_{\tau^-}\cdot u_\tau\,d\tau$ reduces to the usual Itô stochastic contraction term.

1.3. **Stochastic analysis on the path space.** We denote by $\mathbb{P}_{m_0}(M)$ the space of continuous maps $p:[0,1]\to M$, where $M$ is a (compact) Riemannian manifold of dimension $d$, $m_0$ a fixed point in $M$. $\mathbb{P}_{m_0}(M)$ is considered with its natural past filtration and with $\mu$, the Wiener measure, constructed via the fundamental solution of the operator $\partial/\partial\tau - \Delta$, where $\Delta$ is the Laplace-Beltrami operator on $M$.

We consider the stochastic parallel transport of frames, which is the flow of diffeomorphisms on $O(M)$ defined by the following Stratonovich stochastic differential equation:

$$\begin{cases} dr_x(\tau) = \sum_{k=1}^d A_k(r_x)\circ dx^k(\tau) \\ r_x(0) = r_0, \end{cases}$$

with $\pi(r_0) = m_0$. Then $\pi$ sends $\mathbb{P}_{r_0}(O(M))$ into $\mathbb{P}_{m_0}(M)$. The Laplacians on $M$ and on $O(M)$ induce two probability measures; the map $\pi$ realizes an isomorphism between these two probability spaces.

**Definition 1.1.** The map $I : X \to \mathbb{P}_{m_0}(M)$ given by

$$I(x)(\tau) = \pi(r_x(\tau))$$

is called the *Itô map*.

This map is a.s. bijective ([19]) and provides an isomorphism of probability spaces; namely we have

$$\mu = (I)_* \mu_0.$$

**Definition 1.2.** The parallel transport along $p$ is the isomorphism from $T_{p(\tau_0)}(M) \to T_{p(\tau)}(M)$ defined by

$$t^p_{\tau \leftarrow \tau_0} = r_x(\tau) r_x(\tau_0)^{-1},$$

where $x = I^{-1}(p)$.

**Definition 1.3.** A *vector field $z$ along the path $p$* is a *section process* of the tangent bundle of $M$, namely a measurable map $Z_p(\tau) \in T_{p(\tau)}(M)$ defined for $(p, \tau) \in \mathbb{P}_{m_0}(M) \times [0, 1]$.

For a vector field $Z$ along $p$ we shall systematically denote by $z$ the image of $Z$ through the parallelism $\Theta$ given by the parallel transport; more precisely we shall write

$$z_\tau = [\Theta(Z)]_\tau = t^p_{0 \leftarrow \tau}(Z_\tau). \tag{1.8}$$

We define the Itô and the Stratonovich stochastic integrals of an adapted vector field on the path space $Z$, respectively, by

$$\int_0^1 Z \cdot dp = \int_0^1 z^\alpha \, dx_\alpha,$$
$$\int_0^1 Z \circ dp = \int_0^1 z^\alpha \circ dx_\alpha.$$

It is possible to characterize these stochastic integrals without using the parallel transport; they correspond to the limit of the following Riemann sums, when the mesh $|S|$ of the partition $S = \{\sigma_0 = 0 < \sigma_1 < \cdots < \sigma_m = 1\}$ tends to zero:

$$\int_0^1 Z \, dp = \lim_{|S| \to 0} \sum_k \Big( Z_{p(\sigma_{k-1})} \, \Big| \, \exp^{-1}_{p(\sigma_{k-1})}(p(\sigma_k)) \Big)_{T_{p(\sigma_{k-1})}(M)}$$

$$\int_0^1 Z \circ dp = \lim_{|S| \to 0} \frac{1}{2} \sum_k \Big( Z_{p(\sigma_k)} \, \Big| \, \exp^{-1}_{p(\sigma_k)}(p(\sigma_{k+1})) - \exp^{-1}_{p(\sigma_k)}(p(\sigma_{k-1})) \Big)_{T_{p(\sigma_k)}(M)}$$

(for a proof cf. [14]).

In the moving frame type of geometry on the path space, it is natural to consider at the origin the tangent space which corresponds to the one usually associated to Wiener space, namely the Cameron-Martin space. As we have mentioned in the last paragraph, Cameron-Martin vectors are precisely those with respect to which integration by parts can be performed and the corresponding space is dense in $X$. In this perspective, we define

**Definition 1.4.** A *tangent vector field* in $\mathbb{P}_{m_0}(M)$ is a $L^2$-section process $Z$, such that $Z(0) = 0$ and, defining,

$$d^p_\tau Z = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \big( t^p_{\tau \leftarrow \tau + \varepsilon}(Z(\tau + \varepsilon)) - Z(\tau) \big),$$

we have $d^p Z \in L^2$.

On the tangent vector space $T(\mathbb{P}(M))$ we define the Hilbertian norm

$$\|Z\|^2_{T_p(P(M))} = \int_0^1 \|d^p_\tau Z\|^2_{T_{p(\tau)}(M)} d\tau.$$

The parallelism $\Theta$ defined in (1.8) provides a differential 1-form realizing an Hilbertian isomorphism of $T_p(\mathbb{P}_{m_0}(M))$ with the Cameron-Martin space $H^1 = H^1([0,1];\mathbb{R}^d)$ and we have:

$$\frac{d}{d\tau}\Theta(Z) = t^p_{0\leftarrow\tau}(d^p_\tau Z). \tag{1.9}$$

Let $\mathcal{S}(\mathbb{P}_{m_0}(M))$ denote the space of smooth cylindrical functionals on $\mathbb{P}_{m_0}$, namely the functionals $f$ for which there exists a partition of $[0,1]$, $0 \le \tau_1 < \cdots < \tau_m \le 1$ and a smooth function $f$ on $M^m$ such that $F(p) = f(p(\tau_1), \ldots, p(\tau_m))$.

In $M^m$ we have the Riemannian product structure. We define, for $f \in \mathcal{S}(\mathbb{P}_{m_0}(M))$, the following operator:

$$D_\tau F = \sum_{k=1}^m \mathbf{1}_{\tau<\tau_k} t^p_{\tau\leftarrow\tau_k}(\partial_k f). \tag{1.10}$$

The map $\tau \mapsto D_\tau F$ defines a section process (cf. definition 1.3); we introduce the norm

$$\|DF\|^2(p) = \sum_\alpha \int_0^1 (D_{\tau,\alpha}F)^2\, d\tau,$$

where $D_{\tau,\alpha}F = (\, t^p_{0\leftarrow\tau}\, D_\tau\, F \mid \varepsilon_\alpha\,)$, $\{\varepsilon_\alpha\}$ the canonical basis of $\mathbb{R}^d$.

Then, for a tangent vector field $Z$, we define

$$D_Z F = \int_0^1 D_{\tau,\alpha} f \dot{z}^\alpha_\tau\, d\tau. \tag{1.11}$$

In analogy with the Wiener space case, we can consider the "basic vector fields" $\tilde{e}_{\tau,\alpha}(\sigma) = \mathbf{1}_{\tau<\sigma} t^p_{\sigma\leftarrow 0}\varepsilon_\alpha$ and we have $D_{\tau,\alpha} = D_{\tilde{e}_{\tau,\alpha}}$.

The operators $D_{\tau,\alpha}$ may be regarded as forming a "continuous" basis of the tangent space of $\mathbb{P}_{m_0}(M)$.

**Theorem 1.5.** *With respect to the norms $\|Df\|^q_{L^q} = E(\|Df\|^q)$, the operator $D$ is closable in $L^q$. The domain of the operator $D$ is, by definition, the Sobolev space $W_{1,q}(\mathbb{P}_{m_0}(M))$.*

## 2. Differentiability of the Itô parallel transport and intertwining formula

The parallelism we have considered on the path space should allow us to transfer differential calculus on this space to differential calculus on the Wiener space. To do this we are bound to derivate the Itô map, that is, to derivate parallel transport.

**Theorem 2.1.** *Granted the parallelism of $O(M)$, the Jacobian matrix of the flow of diffeomorphisms $r_0 \mapsto r_x(\tau)$ is given by a linear map $J_{x,\tau} = (J^1_{x,\tau}, J^2_{x,\tau}) \in GL(\mathbb{R}^d \times so(d))$ which is defined by the following system of Stratonovich SDEs:*

$$\begin{cases} d_\tau J^1_{x,\tau} = \displaystyle\sum_{\alpha=1}^d (J^2_{x,\tau})^{\cdot}_\alpha \circ dx_\alpha(\tau) \\ d_\tau J^2_{x,\tau} = \displaystyle\sum_{\alpha=1}^d \Omega(J^1_{x,\tau}, \varepsilon_\alpha) \circ dx_\alpha(\tau) \end{cases}$$

*where $(J^2)^{\cdot}_{\alpha}$ denotes the $\alpha$th column of the matrix $J^2$ and $\Omega$ is the curvature tensor of the underlying manifold read on the frame bundle.*

*Proof.* (cf. [10, 14, 20], noting that here the sign of the curvature tensor follows a different convention). Let $x_n$ be a sequence of smooth approximations of the Brownian curve $x$. We consider the $O(M)$-valued map

$$f_n(\tau,t) = r_{x_n+tz}(\tau), \qquad r(\tau_0) = r_0,$$

for $z \in H^1([0,1];\mathbb{R}^d)$, $\tau, t \in [0,1]$. The inverse image by $f_n$ of the differential form of the parallelism is given by

$$f_n^*\theta = \alpha_n\, d\tau + \beta_n\, dt$$
$$f_n^*\omega = \rho_n\, dt$$

where $\alpha_n = \dot{x}_n + t\dot{z}$. Then

$$d(f_n^*\theta) = \left(\frac{\partial \alpha_n}{\partial t} - \frac{\partial \beta_n}{\partial \tau}\right) dt \wedge d\tau$$

and, by the structure equations,

$$f_n^*(d\theta) = \rho_n \alpha_n\, dt \wedge d\tau.$$

Since $d(f_n^*\theta) = f_n^*(d\theta)$, for $t = 0$ we obtain

$$\dot{z} - \frac{\partial \beta_n}{\partial \tau} = \rho_n \dot{x}_n.$$

The second structure equation implies, in an analogous way,

$$\frac{\partial \rho_n}{\partial \tau} = \Omega(\beta_n, \dot{x}_n).$$

The theorem follows from the conditions $\beta_n(0,0) = 0$, $\rho_n(0,0) = 0$ and from a limit theorem for SDEs

$$\begin{cases} d\beta(\tau) = \dot{z}(\tau) - \rho \circ dx(\tau) \\ d\rho(\tau) = \Omega(\beta, \circ dx). \end{cases}$$

Then we take $z = 0$. □

*Remark* 2.1. If one considers a metric connection with torsion on the manifold $M$, the first structure equation must be corrected by the corresponding term and in the last theorem we derive

$$\begin{cases} d\beta(\tau) = \dot{z}(\tau)\, d\tau - \rho \circ dx(\tau) + T(\beta, \circ dx) \\ d\rho(\tau) = \Omega(\beta, \circ dx). \end{cases}$$

**Corollary.** *For $\tau_0 \in [0,1]$ and considering $I_{\tau_0} : X \to M$ the specialization of the Itô map at time $\tau_0$ defined by $x \to \pi(r_x(\tau_0))$, we have*

$$D_\xi I_{\tau_0} = t^p_{\tau_0 \leftarrow 0}(z(\tau_0)),$$

*where*

$$z(\tau_0) = \int_0^1 \hat{J}^x_{\tau_0 \leftarrow \tau} \circ d\xi(\tau),$$

*$\hat{J}$ is the horizontal$\times$horizontal block of matrices $J$ defined in last theorem and*

$$\begin{cases} d\xi = \dot{z}\, d\tau - \rho \circ dx \\ d\rho = \Omega(z, \circ dx). \end{cases}$$

We encounter here a difficulty: the Itô map is not Cameron-Martin differentiable, since the "vector field" $\xi$ is no longer a process of bounded variation! Nevertheless its martingale part is given by an antisymmetric matrix which, by Levy's theorem, implies that Wiener measure is still conserved during the evolution.

If we consider a connection with torsion, an extra martingale term appears, that conserves Wiener measure only if the torsion satisfies the antisymmetric condition

$$T^k(e_i, e_j) = -T^j(e_i, e_k)$$

(Driver's condition).

From this result we see that we have to enlarge the tangent space and that it will not be enough to consider (Cameron-Martin) tangent vector fields. We introduce the following processes:

**Definition 2.2.** A *tangent process* on the Wiener space $X$ is a $\mathbb{R}^d$-valued semi-martingale process $\xi$ defined on $X$ with Itô differential given by

$$d\xi^\alpha(\tau) = a^\alpha_\beta \, dx^\beta(\tau) + c^\alpha \, d\tau,$$

where $a^\alpha_\beta = -a^\beta_\alpha$, $a^\alpha_\beta(0) = 0$, $a^\alpha_\beta$ and $c^\alpha \in L^2[0,1]$.

The *tangent space of* $\mathbb{P}_{m_0}(M)$, that we shall denote by $\tilde{T}(\mathbb{P})$, is the space

$$\{\tilde{\xi}(\tau) = t^p_{\tau\leftarrow 0}\xi(\tau), \ \xi \text{ tangent process on } X\}.$$

Given a smooth cylindrical functional $F(x) = f(x(\tau_1), \ldots, x(\tau_m))$, we define the derivative $D_\xi F$ by

$$D_\xi F = \sum_{k=1}^m \langle d_k f, \xi(\tau_k)\rangle \,. \tag{2.1}$$

The operator $D_\xi$ is closable in $L^2$: this is a consequence of the integration by parts (Theorem 3.1).

**Definition 2.3.** A functional $F$ is called *strongly differentiable* in $L^2$ if

$$F \in \mathrm{Dom}(D_\xi) \quad \forall \text{ tangent process } \xi$$

Which functionals on the Wiener space are actually on the domain of $D_\xi$ or which is the characterization of the closure of this domain is a delicate question. We shall come back to these problems in the next paragraph.

**Theorem 2.4** (Intertwining formula [6])**.** *A scalar-valued functional $F$ defined on the path space is strongly differentiable if and only if $F \circ I$ is strongly differentiable on $X$. We have the intertwining formula*

$$(D_{\xi^*}F) \circ I = D_\xi(F \circ I),$$

*where $\xi$ and $\xi^*$ are related by the equations:*

$$\begin{cases} d\xi = d\xi^* - \rho \circ dx \\ d\rho = \Omega(\xi^*, \circ dx). \end{cases}$$

*Proof.* We consider the following infinitesimal Euclidean motion on the Wiener space

$$[\phi^\xi_t(x)](\tau) = t\xi(\tau) + \int_0^\tau \exp(t\rho) \circ dx,$$

and

$$V^\xi_t = I \circ \phi^\xi_t \circ I^{-1};$$

derivating in $t = 0$,

$$\frac{d}{dt}\bigg|_{t=0} V_t^\xi = \pi'(r_x(\tau)) \cdot \frac{d}{dt}\bigg|_{t=0} r_{\phi_t^\xi(x)}(\tau) = r_x(\tau)\, \xi^*(\tau),$$

and the result follows from last corollary. □

*Remark* 2.2. For a Driver-type connection we have to replace the last equations by

$$\begin{cases} d\xi = d\xi^* - \rho \circ dx + T(\xi^*, \circ dx), \\ d\rho = \Omega(\xi^*, \circ dx), \end{cases}$$

where $T$ is read on the frame bundle, $T_r(u,v) = r^{-1} T(ru, rv)$.

At this stage one could think we are dealing with two different notions of derivative on the path space, the one defined in paragraph 1.3 and the one that naturally follows from the above results, namely, for $F \in \mathcal{S}(\mathbb{P}_{m_0}(M))$, $F(p) = f(p(\tau_1), \ldots, p(\tau_m))$,

$$D_{\tau,\alpha} F(p) = \frac{d}{dt}\bigg|_{t=0} F(V_t^{\tilde{e}_{\tau,\alpha}}(p)),$$

the limit being taken in $L^p(\mu)$ with $p > 1$.

In fact both notions coincide; we have:

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=0} F\big(V_t^{\tilde{e}_{\tau,\alpha}}(p)\big) &= \sum_{k=1}^m \partial_k f(p) \cdot \frac{d}{dt}\bigg|_{t=0} V_t^{\tilde{e}_{\tau,\alpha}}(\tau_k) \\ &= \sum_{k=1}^m \partial_k f(p) \cdot \big(\mathbf{1}_{\tau<\tau_k} t^p_{\tau_k \leftarrow 0} \varepsilon_\alpha\big) \\ &= D_{\tau,\alpha} F. \end{aligned}$$

The next result gives a formula for the derivation of the parallel transport on the path space.

**Theorem 2.5.** *For fixed $\tau_0 \in [0,1]$ and denoting $\Phi(p) = t^p_{\tau_0 \leftarrow 0} r_0$, the derivative of $\Phi$ can be expressed in the parallelism of $O(M)$ as:*

$$\langle D_Z \Phi, \theta \rangle = z(\tau_0) \quad \big(+ \int_0^\tau T(z, \circ dx)\big),$$

$$\langle D_Z \Phi, \omega \rangle = \int_0^{\tau_0} \Omega(z, \circ dx).$$

*Proof.* Derivating on the path space with respect to a tangent vector field $Z$ means, by the intertwining formula, derivating with respect to a tangent process

$$\begin{cases} d\xi = \dot{z}\, d\tau - \rho \circ dx \quad \big(+T(z, \circ dx)\big) \\ d\rho = \Omega(z, \circ dx) \end{cases}$$

the functionals pulled back to the Wiener space through the Itô map. □

We have obtained the derivation of the parallel transport with a short proof, by transferring the result to the Wiener space. This result can also be proved by a more direct geometric analysis, an approach that may have the advantage of a more intuitive argument, but requires a very delicate approximation procedure. Here we just sketch the main argument.

We take cylindrical approximations of the functional $t^p_{\tau_0 \leftarrow 0} r_0$ obtained by parallel transporting along piecewise minimizing geodesics $\gamma_n$ based on points $\{p(\tau_1), \ldots, p(\tau_n)\}$ of the manifold $M$ and converging to Brownian motion on $M$. For such geodesics to be well defined one must place ourselves inside a ball of radius less than the radius of injectivity: that is, one must consider a cutoff function procedure together with the approximation one (we refer to [6], paragraph II-4 for the development of such techniques).

We want to differentiate parallel transport on the path space. Working with a normal chart centered at a fixed point $p(\tau_k)$, this means that we want to compare in an infinitesimal way parallel translation along the geodesics going from $p(\tau_{k-1})$ to $p(\tau_k)$ and from $p(\tau_k)$ to $p(t_{k+1})$ to parallel translation when $p(\tau_k)$ is perturbed in the direction we want to consider. So, modulo the bracket of the vector fields involved, we are considering a loop going from $p(\tau_{k-1})$ to $p(\tau_{k+1})$ and back. To compute parallel transport along this loop is precisely to compute the holonomy of the curve in Differential Geometry, which means integrating the curvature along the path ([17]). The integrals converge at the end to Stratonovich integrals with respect to Brownian motion.

## 3. The space of tangent processes

We consider the theory of anticipative integrals according to Nualart-Zakai-Pardoux, following reference [22]. Given a scalar valued process $u_\tau$, its Skorohod and Skorohod-Stratonovich integrals, that we denote, respectively, by $\int_0^1 u\,dx$ and $\int_0^1 u \circ dx$, are defined as the limit of the Riemannian sums (1.3) and (1.6), when they exist.

Let $\xi_\tau$ be a tangent process, namely a process satisfying the stochastic differential equation

$$d\xi^\alpha(\tau) = a^\alpha_\beta\, dx^\beta(\tau) + c^\alpha\, d\tau$$

(cf. definition 2.2).

**Theorem 3.1** (Integration by parts). *For every smooth cylindrical functional $F$ we have*

$$E(D_\xi F) = E(F\int_0^1 c_\alpha\, dx_\alpha),$$

*where $D_\xi F$ was defined in* (2.1).

*Proof.* The martingale part of the Itô representation of $\xi$ defines a measure preserving isomorphism on the Wiener space. □

We define the Skorohod and the Skorohod-Stratonovich integrals of a process $u_\tau$ relatively to a tangent process $\xi$ as the limit of the sums

$$\sum_k \mathcal{M}_k(u)\cdot(\xi(\tau_{k+1}-\xi(\tau_k))$$

and

$$\sum_k E^{{\delta_k}^c}(\mathcal{M}_k(u))\cdot(\xi(\tau_{k+1}-\xi(\tau_k)),$$

where $\mathcal{M}_k(u)$ was defined in (1.3) and $E^{{\delta_k}^c}$ denotes the conditional expectation constituted by averaging relatively to the $\sigma$-field generated by $x(\tau)-x(\tau_k)$, $\tau\in\delta_k=[\tau_k,\tau_{k+1}]$.

**Theorem 3.2.** *Assume that* $f \in W_2^p(X)$ $\forall p > 1$ *and that* $a^{\cdot}_{\cdot} \in L^p(X; L^2[0,1])$ $\forall p \geq 1$, *a is adapted and* $a^\alpha_\beta = -a^\beta_\alpha$. *If one of the two stochastic integrals below exist, then the other exists as well and*

$$\int_0^1 (D_{\tau,\alpha} f) \cdot d\xi^\alpha(\tau) = \int_0^1 (D_{\tau,\alpha} f) \circ d\xi^\alpha(\tau),$$

*where* $\xi^\alpha(\tau) = \int_0^\tau a^\alpha_\beta \, dx^\beta$.

*Proof.* The difference between the two integrals is given by the limit of the following sums:

$$\sum_k (E^{\delta_k{}^c}(\mathcal{M}_k(D_{\cdot,\alpha} f)) - \mathcal{M}_k(f))(\xi^\alpha(\tau_{k+1}) - \xi^\alpha(\tau_k));$$

using the Clark-Ocone formula, this expression is equal to

$$-\sum_k \left( \mathcal{M}_k \int_{\sigma_k}^{\sigma_{k+1}} E^{\mathcal{P}_\lambda}(D^2_{(\lambda,\gamma),(\cdot,\alpha)} f) \, dx^\gamma(\lambda) \right) \cdot (\xi^\alpha(\tau_{k+1}) - \xi^\alpha(\tau_k))$$

and the limit when the mesh of the partition goes to zero is equal to the limit of

$$-\sum_{k,\gamma} \int_{\tau_k}^{\tau_{k+1}} \mathcal{M}_k(D^2_{\lambda,\gamma),(\cdot,\alpha)} f) a^\alpha_\gamma(\lambda) \, d\lambda$$

which is equal to zero by the symmetry of the second derivatives and the antisymmetry of $a^{\cdot}_{\cdot}$. □

Generalizing the corresponding representation formula for derivatives along Cameron-Martin vector fields (cf. (1.2)), we have the following:

**Theorem 3.3.** *Let* $\xi$ *be a tangent process such that* $a^{\cdot}_{\cdot}$ *satisfies the assumptions of theorem 3.2 and, furthermore, that* $a^\alpha_\beta \in W_{1,p}(X)$ $\forall p \geq 1$, *and* $\int_0^1 \|c(\tau)\|_{L^2(X)} \, d\tau < +\infty$. *Then* $W_2^q \subset$ *Dom*$(D_\xi)$ $\forall q > 1$ *and we have*

$$D_\xi f = \sum_\beta \int_0^1 \left( \sum_\alpha a^\alpha_\beta \, D_{\tau,\alpha} f \right) \cdot dx^\beta(\tau) + \int_0^1 c_\alpha \, D_{\tau,\alpha} \, f \, d\tau.$$

For a proof of this result we refer to [6] and to the appendix in [8].

We may use the representation of last theorem to derive a formula for the derivative of a stochastic integral with respect to a tangent process. These formulae for derivations with respect to Cameron-Martin space valued processes were obtained in [25].

**Theorem 3.4** ([6])**.** *Let* $\xi$ *be a tangent process with coefficients satisfying the assumptions of theorem 3.3. Let* $u$ *be an adapted process such that, for some* $p > 1$, $\int_0^1 \|u(\tau)\|_{2,p} \, d\tau < +\infty$. *The derivative of the Itô stochastic integral of* $u$ *is given by:*

$$D_\xi \int_0^1 u \cdot dx = \int_0^1 D_\xi u \cdot dx + \int_0^1 u \cdot d\xi.$$

Also in [6] we have derived a corresponding formula for the derivation of Stratonovich integrals. Under suitable assumptions that ensure the existence of such integrals, it reads:

$$D_\xi \int_0^1 u \circ dx = \int_0^1 D_\xi u \circ dx + \int_0^1 u \circ d\xi.$$

Tangent processes have the same regularity (in time) as the Wiener process; therefore it is not possible to extend to the space of tangent processes the $H^1$

metric. Considering $H^{\frac{1}{2}-\varepsilon}$ metrics gives rise to serious difficulties, namely in the definition of corresponding metric (for instance, Levi-Civita) connections (cf. [9]).

## 4. Integration by parts on the path space

A formula of integration by parts on the path space was first derived by Bismut [2]. There are different proofs and approaches to this result: we refer to [1, 10, 14]. In this paragraph we derive integration by parts on the path space via transferring the result to the Wiener space and using the intertwining theorem.

Let $Z$ be a tangent vector field on the path space. From the results in paragraph 2 we have, for cylindrical functionals $F$,

$$E_\mu(D_Z F) = E_{\mu_0}(D_\xi(F \circ I)),$$

where $d\xi = \dot{z}\, d\tau - \rho \circ dx$, $d\rho = \Omega(z, \circ dx)$ and $Z_\tau = t^p_{\tau\leftarrow 0} z_\tau$. On the Wiener space, we have

$$E_{\mu_0}(D_\xi(F \circ I)) = E_{\mu_0}((F \circ I)\delta(\xi)).$$

The process $\xi$ is a tangent process whose bounded variation part is equal to $\dot{z}\, d\tau + \frac{1}{2}\, d\rho \cdot dx$. From the equations of $\rho$, $d\rho \cdot dx = \mathrm{Ricci}(z)\, d\tau$, where $\mathrm{Ricci}(z)_\tau = t^p_{\tau\leftarrow 0} \circ \mathrm{Ricci}_{p(\tau)}\, Z \circ t^p_{\tau\leftarrow 0}$. We have, therefore,

**Theorem 4.1** (Bismut integration by parts formula)**.** *For a cylindrical function $F$ on the path space and $Z$ an adapted vector field such that $E\int_0^1 |d^p_\tau Z|^2\, d\tau < +\infty$,*

$$E(D_Z F) = E\biggl((F \circ I) \int_0^1 \Bigl[\dot{z} + \frac{1}{2}\,\mathrm{Ricci}(z)\Bigr]\, dx\biggr).$$

From this theorem it follows that $D$ is a closable operator from $L^p(\mathbb{P}_{m_0}(M))$ to the space

$$\Bigl\{z : Z \text{ tangent vector field, } \|Z\|_p^p = E\Bigl(\int_0^1 \|d^p_\tau Z\|^2\, d\tau\Bigr)^{p/2} < +\infty\Bigr\}.$$

We remark that, when the connection considered on the manifold is of Driver type, an extra term appears, namely

$$\frac{1}{2}\, dT \cdot dx, \qquad \text{where } dT = T(z, \circ dx).$$

In this case we derive the following integration by parts formula:

$$E(D_Z F) = E\biggl((F \circ I) \int_0^1 \Bigl[\dot{z} + \frac{1}{2}\,\mathrm{Ricci}(z) + \hat{T}(z)\Bigr]\, dx\biggr),$$

where $\hat{T}(z) = \sum_{\alpha=1}^d (\nabla_{e_\alpha} T)(z, e_\alpha)$, a result which is due to Driver [10].

We have only considered adapted vector fields $Z$. A natural question is what happens if $Z$ is anticipative and whether in this case the divergence could be simply written, in analogy with what happens in the Wiener space, as

$$\delta(z) = \int_0^1 \Bigl(\dot{z} + \frac{1}{2}\,\mathrm{Ricci}(z)\Bigr) \cdot dx,$$

where the stochastic integral would be interpreted in the sense of Skorohod. The answer is no; another term involving derivatives of $Z$ and a stochastic integral of the curvature tensor appears. The corresponding formula was obtained in [7], where we have developed a computational technique of decomposition of the processes in their "continuous coordinates" expressed on the basic vector fields.

We have, for non necessarily adapted tangent vector fields $Z$, and for cylindrical functions $F$ on the path space,

$$E(D_Z F) = E\Big((F \circ I)\Big[\int_0^1 \dot{z}\, dx + \int_0^1 \dot{z}_\tau \cdot g_\tau \, d\tau - \int_0^1 D_{q_\tau} \cdot z_\tau \, d\tau\Big]\Big), \tag{4.1}$$

where

$$g_{\tau,\alpha} = \frac{1}{2}\int_\tau^1 (\mathrm{Ricci}\,)^\alpha_\beta \, dx^\beta$$

and

$$q_{\tau,\alpha}(\sigma) = \mathbf{1}_{\tau<\sigma} \int_\tau^\sigma \Big[\int_\tau^\lambda \Omega(\circ dx, \varepsilon_\alpha)\Big] \circ dx(\lambda).$$

The integration by parts formula (4.1) holds under suitable regularity assumptions on $Z$ that ensure the definition of the anticipative stochastic integrals involved (cf. [7]).

## 5. Structural equations of the path space

In this section we compute the bracket of two constant vector fields, namely

$$U(p)_\tau = t^p_{\tau\leftarrow 0} u_\tau, \qquad V(p)_\tau = t^p_{\tau\leftarrow 0} v_\tau,$$

where $u, v$ are non random.

Let $F(p) = f(p(\tau_1), \ldots, p(\tau_m))$ be a smooth cylindrical functional; denote by $\tilde{F}$ the lift of $F$ to $[O(M)]^m$, namely

$$\tilde{F}(r_p(\tau_1), \ldots, r_p(\tau_m)) = F(\pi(r_p(\tau_1)), \ldots, \pi(r_p(\tau_m))),$$

and by $\partial_{i,\alpha}\tilde{F}$ the derivative of $\tilde{F}$ in the coordinate $r_p(\tau_i)$ and in the direction of the horizontal vector field $A_\alpha$:

$$\partial_{i,\alpha}\tilde{F} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \tilde{F}(r_p(\tau_1), \ldots, r_p(\tau_i) + \varepsilon A_\alpha, \ldots, r_p(\tau_m)).$$

Then the following equality holds:

$$D_U f = \sum_{i=1}^m u^\alpha(\tau_i)\, \partial_{i,\alpha}\tilde{F}(r_p(\tau_1), \ldots, r_p(\tau_m)),$$

and we have

$$\begin{aligned} D_V D_U F &= \sum_{i,j} v^\beta(\tau_j)\, u^\alpha(\tau_i)\, \partial_j(\partial_i \tilde{F} \cdot r^\alpha(\tau_i)) \cdot r^\beta(\tau_j) \\ &= \sum_{i,j} v^\beta(\tau_j)\, u^\alpha(\tau_i)\, \big[\partial_{j,\beta}\, \partial_{i,\alpha}\tilde{F} + \partial_{i,\gamma}\tilde{F}(\partial_{j,\beta} r^\alpha(\tau_i))^\gamma\big]. \end{aligned}$$

When $i \neq j$, $\partial_i$ and $\partial_j$ commute; when $i = j$,

$$\partial_{i,\alpha}\, \partial_{i,\beta} - \partial_{i,\beta}\, \partial_{i,\alpha} = \partial_{i,[A_\alpha, A_\beta]}.$$

Since $A_\alpha$ and $A_\beta$ are horizontal vector fields, $[A_\alpha, A_\beta]$ is vertical; on the other hand, $\tilde{F}$ only depends on $\pi(r)$, therefore this term vanishes. It remains to consider the term corresponding to the derivative of the parallel transport

$$\sum_j v^\beta(\tau_j)\,\partial_{j,\beta} r^\alpha(\tau_i) = D_V r^\alpha(\tau_i)$$
$$= \int_0^1 \dot v^\beta(\tau)\left(\mathbf{1}_{\tau<\tau_i}\int_\tau^{\tau_i}\Omega_{\beta,\lambda,\alpha}\circ dx^\lambda\right)d\tau$$
$$= \int_0^\tau v^\beta(\tau)\,\Omega_{\beta\lambda\alpha}\circ dx^\lambda.$$

We have, therefore,

$$(D_U D_V - D_V D_U)F$$
$$= \sum_{i=1}^m \left\langle \partial_{s,\gamma} f,\, v^\beta(\tau_i)\int_0^{\tau_i}\Omega_{\alpha\lambda\beta}u^\alpha\circ dx^\lambda - u^\alpha(\tau_i)\int_0^{\tau_i}\Omega_{\beta\lambda\alpha}v^\beta\circ dx^\lambda\right\rangle,$$

from which we deduce the following

**Theorem 5.1.** *The bracket of two constant tangent vector fields $U$ and $V$ on the path space is given by the following expression in the parallelism of the moving frame:*

$$[u,v]_\tau = Q_u v - Q_v u, \qquad \textit{where } Q_u(\tau) = \int_0^\tau \Omega(u,\circ dx).$$

**Corollary.** *The bracket of two constant (Cameron-Martin) tangent vector fields is no longer a Cameron-Martin vector field.*

*Proof.* In differential form, the bracket is given by

$$d_\tau[u,v] = \Omega(u,v)\circ dx + [Q_u\dot v - Q_v\dot u]\,d\tau.$$

□

We encounter here a new phenomena, the non-closure of the tangent space consisting of tangent vector fields under the bracket. We also encounter a new reason to enlarge the tangent space by considering tangent processes.

In particular, we have, for basic vector fields,

$$[\tilde e_{\tau,\alpha},\tilde e_{\sigma,\beta}]_{(\rho,\gamma)} = \mathbf{1}_{\sigma<\rho}\int_\tau^\rho \Omega^\gamma_{\alpha,\lambda,\beta}\circ dx^\lambda - \mathbf{1}_{\tau<\rho}\int_\sigma^\rho \Omega^\gamma_{\beta,\lambda,\alpha}\circ dx^\lambda.$$

We remark that, even inside the same time interval, there is no possibility of simplifying the curvature terms. In fact, and unless we are in a flat manifold, the diffusion term of the bracket does not vanish.

Let us consider a map $A:\mathbb{P}_{m_0}(M)\to \mathrm{End}(H^1(\mathbb{R}^d))$ which is invertible and a new parallelism defined by $\tilde\Theta = A\circ\Theta$. For $u_1,u_2\in H^1$, let $\tilde u_i = (A^{-1})u_i$, $\tilde V_i = \Theta^{-1}(\tilde u_i)$, $i=1,2$; we compute $D_{V_3} = D_{V_1}D_{V_2} - D_{V_2}D_{V_1}$ and identify $V_i$ with $v_i$ through the parallelism $\Theta$, obtaining

$$v_3 = [v_1,v_2] + (D_{v_1}A^{-1})u_2 - (D_{v_2}A^{-1})u_1.$$

Since the last terms are Cameron-Martin vector fields, we see that a change of metric on the path space does not change the fact that the bracket produces a true tangent process.

Nevertheless a very interesting phenomena is that the tangent processes (the "enlarged" tangent space) do form a Lie algebra: the bracket of two tangent processes is again a tangent process. The result was shown in [6] and [11].

**Theorem 5.2.** *Given two smooth tangent processes* $\xi_1$ *and* $\xi_2$ *on* $\mathbb{P}_{m_0}(M)$*, there exists a tangent process* $\xi_3$ *such that, denoting* $B = D_{\xi_1}D_{\xi_2} - D_{\xi_2}D_{\xi_1}$*, we have*

$$BF = D_{\xi_3}F.$$

## 6. Riemannian connections

As we have recalled in 1.1, in finite dimensions, the Levi-Civita connection, the only Riemannian connection which is torsion free, is determined by the structure equations. As we have computed those on the path space, we can also consider the corresponding Levi-Civita connection.

For $U_i = t^p_{\tau\leftarrow 0}u_i$, $u_i \in H^1$, $i = 1, 2, 3$, and identifying again vector fields on the path space with the corresponding Cameron-Martin processes through the parallelism, a Riemannian connection without torsion (Levi-Civita connection), $\tilde{\nabla}_{U_1}U_2$, will be defined by

$$(\tilde{\nabla}_{u_1}u_2 \mid u_3) = \frac{1}{2}\big(([u_1, u_2] \mid u_3) - ([u_2, u_3] \mid u_1) + ([u_3, u_1] \mid u_2)\big).$$

Using the expression for the bracket,

$$([u_i, u_j] \mid u_k) = \int_0^1 \dot{u}_k \Omega(u_i, u_j)(\circ dx) + \int_0^1 \dot{u}_k [Q_{u_i}\dot{u}_j - Q_{u_j}\dot{u}_i]\, d\tau. \tag{6.1}$$

Integrating by parts

$$\int_0^1 \dot{u}_1 \Omega(u_2, u_3)(\circ dx) = \int_0^1 \Omega^\gamma_{\alpha\beta\lambda} u_2^\alpha u_3^\beta \dot{u}_1^\gamma \circ dx^\lambda,$$

we obtain

$$\int_0^1 \left[\int_\tau^1 \Omega^\gamma_{\alpha\beta\lambda} u_2^\alpha \dot{u}_1^\gamma \circ dx^\lambda\right] \dot{u}_3^\beta \, d\tau.$$

The sum of the contributions of the Stratonovich integrals in expression (6.1) is equal to

$$\frac{1}{2}\int_0^1 \Omega(u_1, u_2)(\circ dx)\dot{u}_3 + \frac{1}{2}\int_0^1 \left[\int_\tau^1 \Omega(\dot{u}_1, \circ dx)(u_2) + \Omega(\dot{u}_2, \circ dx)(u_1)\right] \dot{u}_3 \, d\tau.$$

Using the antisymmetry of the matrices $Q$ we obtain:

**Theorem 6.1.** *The Levi-Civita covariant derivative* $\tilde{\nabla}_{u_1}u_2$ *of two constant tangent vector fields has the following expression in the parallelism of the moving frame:*

$$d_\tau(\tilde{\nabla}_{u_1}u_2) = \frac{1}{2}\Omega(u_1, u_2)(\circ dx) + \left[Q_{u_1}\dot{u}_2 + \frac{1}{2}\int_\tau^1 \Omega(\dot{u}_1, \circ dx)(u_2) + \frac{1}{2}\int_\tau^1 \Omega(\dot{u}_2, \circ dx)(u_1)\right] d\tau.$$

We remark that the expression obtained is a tangent process with an anticipative bounded variation part.

Various other connections can be defined on the path space. We shall work in the sequel with a particular one, that we call the Markovian connection.

**Definition 6.2.** For two constant tangent vector fields $U_1$, $U_2$, the Markovian covariant derivative is defined by

$$[d^p_\tau \nabla_{U_1}U_2](p) = D_{U_1}[(\exp^{-1}_{p(\tau)})_*(d^p_\tau U_2)].$$

This expression is Markovian in the sense that $d^p_\tau[\nabla_{U_1}U_2]$ depends only upon $d^p_\tau U_2$ and $U_1(\tau)$.

**Theorem 6.3.** *The Markovian connection is expressed in the parallelism by*

$$\frac{d}{d\tau}(\nabla_{u_1} u_2) = Q_{u_1} \dot{u}_2,$$

*for* $u_1, u_2 \in H^1$.

*Proof.* Since $U_2(p)(\tau) = t^p_{\tau \leftarrow 0} u_2(\tau)$, we have

$$\frac{d}{d\tau}[\Theta(\nabla_{U_1} U_2)] = \langle \omega, D_{U_1}(t^p_{\tau \leftarrow 0}) \rangle * \dot{u}_2(\tau)$$

and the theorem of derivation of the parallel transport (section 2) gives the result. □

We introduce the localization of the covariant derivative by the definition

$$\nabla_{\tau,\alpha} Z = \nabla_{\tilde{e}_{\tau,\alpha}} Z.$$

For a tangent vector field $Y$, we have

$$\nabla_Y Z = \sum_{\alpha=1}^{d} \int_0^1 \nabla_{\tau,\alpha} Z d^p_{\tau,\alpha} Y \, d\tau. \tag{6.2}$$

The Christoffel symbols of the Markovian connection are defined by

$$\nabla_{\tau,\alpha}(\tilde{e}_{\sigma,\beta}) = \Gamma^\gamma_{\alpha\beta}(\tau,\sigma) \, \tilde{e}_{\sigma,\beta},$$

where

$$\Gamma^\gamma_{\alpha\beta}(\tau,\sigma) = \mathbf{1}_{\tau<\sigma} \int_\tau^\sigma \Omega^\gamma_{\alpha\lambda\beta} \circ dx^\lambda. \tag{6.3}$$

**Theorem 6.4.** *The Markovian connection is Riemannian. Its torsion is given by*

$$T^\gamma(\tilde{e}_{\tau,\alpha}, \tilde{e}_{\sigma,\beta})(s) = -\mathbf{1}_{\tau\vee\sigma\le s} \int_{\tau\vee\sigma}^s \Omega^\gamma_{\alpha\beta\lambda} \circ dx^\lambda.$$

*Proof.* Let $\hat{U}_i(\sigma)$, $i = 1, 2$, denote the vector fields $d_\tau U_i(\sigma)$ read in the normal chart at $\exp^{-1}_{p(\tau)}$ and $g_{ij}$ the metric tensor of $M$ read in this normal chart. The fact that the derivatives of $g_{ij}$ vanish at the origin, implies that

$$D_Y[g_{ij} \hat{U}_1{}^i U_2{}^j] = g_{ij}[D_Y \hat{U}_1{}^i] U_2{}^j + g_{ij} U_1{}^i [D_Y U_2{}^j]$$

and

$$D_Y \hat{U}_1{}^i(\sigma) = d^p_\tau[\nabla_Y U_i(\sigma)]$$

The expression for the torsion follows from the structural equations. □

## 7. Weitzenböck formulae

7.1. **Energy identities and curvature.** As we have recalled in section 1.2 energy identities are fundamental in Stochastic Analysis. They are at the basis of the definition of stochastic integrals of adapted processes and they allow to derive estimates for anticipative stochastic integrals. On the Wiener space the energy equality for anticipative integrals is:

$$E\left|\int_0^1 u_\tau \, dx(\tau)\right|^2 = E\int_0^1 |u_\tau|^2 \, d\tau + E\int_0^1\int_0^1 D_\tau u_\sigma \cdot D_\sigma u_\tau \, d\tau \, d\sigma$$

In Differential Geometry formulae of the type

$$dd^* + d^*d = -\Delta + Ric,$$

where $d^*$ denote the adjoint of the exterior derivative with respect to the Riemann measure $dm$ and $Ric$ is the Ricci tensor associated with the Levi-Civita connection $\nabla$ are known under the name of Weitzenböck formulae. For a metric connection with torsion, one has

$$dd^* + d^*d = -\Delta + Ric + \hat{T},$$

where $\hat{T}(e_j) = \sum_{i=1}^d (\nabla_{e_i} \cdot T)(e_j, e_i)$. If we consider Weitzenböck formulae (with respect to Levi-Civita connection) on 1 forms $\omega_z$ ($z$ denotes the dual correspondent vector field) we obtain

$$\int |d^*z|^2\, dm + \int |d\omega_z|^2\, dm = \int |\nabla z|^2\, dm + \int \langle Ric\, z, z\rangle\, dm,$$

which is equivalent to

$$\int |d^*z|^2\, dm = \sum_{i,j} \int (\nabla_{e_i} z \mid e_j)(\nabla_{e_j} z \mid e_i)\, dm + \int \langle Ric\, z, z\rangle\, dm,$$

where $\{e_i\}$ denotes an orthonormal basis of the tangent space.

This corresponds to the energy identity written in the Wiener space with respect to the underlying Gaussian measure. We may say, in an equivalent way, that the Ricci tensor of the Wiener space is equal to the identity. This result was obtained by Shigekawa in [23].

If $\Box = dd^* + d^*d$ denotes the Rham-Hodge operator on forms of degree one, the semigroup $e^{-t\Delta} f$ satisfies

$$\frac{\partial}{\partial t}(de^{-t\Delta} f) = dd^* de^{-t\Delta} f = \Box(de^{-t\Delta} f),$$

since $ddu = 0$. The problem of estimating the commutator between $de^{-t\Delta}$ and $e^{-t\Delta} d$ reduces to estimating the commutator between the operators $\Delta$ and $\Box$ on differential forms (cf. [1] for a development of this point of view).

On the Wiener space Mehler's formula gives an explicit representation of the semigroup associated to the Ornstein-Uhlenbeck operator $\mathcal{L} f = -\delta df$. The commutation relation reads

$$d(e^{-t\mathcal{L}} f) = e^{-t}(e^{-t\mathcal{L}} df)$$

and is at the basis of Meyer's inequalities (cf. [21]).

7.2. **First order commutation relations.** An energy identity, as we have seen in paragraph 1.2, follows from a commutation relation between derivatives and divergences which means, in the case of adapted vector fields, between derivatives an (Itô) stochastic integrals. We are therefore interested in studying such relations on the path space.

We have the following

**Theorem 7.1.** *Given an adapted tangent vector field $Z$ such that the process $\dot{z}$ satisfies $\int_0^1 \|\dot{z}(\tau)\|_{2,p}\, d\tau < +\infty$ for some $p > 1$ we have*

$$D_{\tau,\alpha} \int_0^1 d^p_\sigma Z \cdot dp(\sigma) = \int_0^1 d^p_\sigma (\nabla_{\tau,\alpha} Z) \cdot dp(\sigma) + d^p_{\tau,\alpha} Z - \frac{1}{2} \int_\tau^1 (Ric\, \dot{z})^\alpha\, d\tau.$$

*Proof.* We start from the characterization of the Itô integral on the path space,

$$\int_0^1 d^p_\sigma Z \cdot dp(\sigma) = \int_0^1 \dot{z}_\alpha\, dx^\alpha$$

and we observe that, by the intertwining theorem, the derivation $D_{\tau,\alpha}$ corresponds, on the Wiener space, to the derivation with respect to the tangent process

$$\xi_{\tau,\alpha}(\sigma) = \mathbf{1}_{\tau<\sigma}\,\varepsilon_\alpha + \int_\tau^\sigma \left( \int_\tau^s \Omega(\circ dx, \varepsilon_\alpha) \right) \circ dx(s).$$

By theorem 3.4,

$$D_\xi \int_0^1 \dot z \cdot dx = \int_0^1 (D_\xi\, \dot z) \cdot dx + \int_0^1 \dot z \cdot d\xi.$$

We have $\dot z_\sigma = r^{-1}(d^p_\sigma Z)$ and we derivate parallel transport by making the derivative at the point $p(\tau)$ and using the normal chart centered at this point. From the formulae of the derivative of the parallel transport and the definition of the Markovian connection it follows that

$$\left[D_{\tau,\alpha}\,\dot z\right]^\beta_\sigma = \left[r^{-1}(d^p_\sigma \nabla_{\tau,\alpha} Z)\right]^\beta - \Gamma_{\alpha\gamma}^{\;\beta}(\tau,\sigma)\,\dot z^\gamma_\tau,$$

where $\Gamma$ denotes the Christoffel symbols defined in (6.3).

Concerning the second term,

$$\begin{aligned}\int_0^1 \dot z \cdot d\xi = \dot z^\alpha(\tau) - & \int_\tau^1 \left( \int_\tau^\sigma \Omega(\circ dx, \varepsilon_\alpha) \right) \dot z(\sigma) \cdot dx(\sigma) \\ & - \frac{1}{2} \int_0^1 \sum_{\beta,\gamma} \dot z^\gamma(\sigma)\, d\Gamma_{\alpha\gamma}^{\;\beta}(\tau,\sigma) \cdot dx^\beta(\sigma).\end{aligned}$$

Since $d\Gamma_{\alpha\gamma}^{\;\beta}(\tau,\sigma) \cdot dx^\beta(\sigma) = \Omega^\beta_{\alpha\beta\gamma}\, d\tau$, we obtain the result. □

We observe that the Markovian connection appears naturally when dealing with first order commutation relations on the path space.

An analogous formula for derivating Stratonovich stochastic integrals may be derived. Under suitable assumptions on the tangent vector field $Z$, it reads

$$D_{\tau,\alpha} \int_0^1 d^p_\alpha Z \circ dp(\sigma) = \int_0^1 d^p_\sigma(\nabla_{\tau,\alpha} Z) \circ dp(\sigma) + d^p_{\tau,\sigma} Z \tag{7.1}$$

(cf. [6]).

7.3. **A first result for adapted tangent vectors.** Using Bismut's characterization of the divergence in terms of stochastic integrals, together with the commutation relations of the last paragraph, we may derive a first energy identity for adapted vector fields on the path space.

A differential form of degree $p$ on the path space is given by a functional $\rho \in W_{rq}\left(\mathcal{P}_{m_0}(M); [H^1]^{\wedge^p}\right)$.

**Definition 7.2.** Given a form of degree 1 its coboundary is defined by

$$\langle d\rho, Z_1 \wedge Z_2 \rangle = D_{Z_1}(\langle \rho, Z_2 \rangle) - D_{Z_2}(\langle \rho, Z_1 \rangle) - \langle \rho, [Z_1, Z_2] \rangle\,.$$

Granted the Hilbertian structure of the underlying tangent space, differential forms of degree 1 may be identified with linear functionals on the space of tangent vector fields.

Let $Z$ be a tangent vector field on the path space. We have

$$\begin{aligned}
&E\big(\delta(Z)\big)^2\\
=&E\big(D_Z(\delta Z)\big)\\
=&E\left(\sum_\beta\int_0^1 d^p_{\sigma,\beta}\,Z\left[D_{\alpha,\beta}\int_0^1\left(d^p_\tau Z+\frac12(Ric\,Z)_\tau\right)\cdot dp(\tau)\right]d\sigma\right)\\
=&E\left(\sum_\beta\int_0^1 d^p_{\sigma,\beta}\,Z\left[\int_0^1 d^p{}_\tau(\nabla_{\sigma,\beta}\,Z)+\frac12\nabla_{\sigma,\beta}(Ric\,Z)_\tau\cdot dp(\tau)\right]d\sigma\right)+I,
\end{aligned}$$

where

$$\begin{aligned}
I=&\,E\|Z\|^2_{H^1}+\frac12 E(Ric\,Z\mid Z)_{H^1}\\
&-\frac12 E\left(\sum_\beta\int_0^1 d^p_{\sigma,\beta}\,Z\left(\int_\sigma^1 Ric\,(d^p_\tau\,Z)+Ric\,(Ric\,Z)^\beta_\tau\,d\tau\right)\right).
\end{aligned}$$

Then we decompose

$$\nabla_{\sigma,\beta}(Ric\,Z_\tau)=\big[\nabla_{\sigma,\beta}Ric\big]\,Z_\tau+Ric\,\big[\nabla_{\sigma,\beta}\,Z\big]_\tau.$$

The first term gives rise to a stochastic integral which is again a divergence, namely

$$\begin{aligned}
E\left(\sum_\beta\int_0^1\delta(\nabla_{\sigma,\beta}\,Z)d^p_{\sigma,\beta}\,Z\,d\sigma\right)&=E\left(\sum_\beta\int_0^1 D_{\nabla_{\sigma,\beta}\,Z}\big(d^p_{\sigma,\beta}\,Z\big)d\sigma\right)\\
&=E\left(\sum_{\alpha,\beta}\int_0^1\!\!\int_0^1 d^p_{\tau,\alpha}(\nabla_{\sigma,\beta}\,Z)D_{\tau,\alpha}\,(d^p_{\sigma,\beta}\,Z)d\tau\,d\sigma\right).
\end{aligned}$$

Now $D_{\tau,\sigma}(d^p_{\sigma,\beta}\,Z)=d^p_{\sigma,\beta}(\nabla_{\tau,\alpha}\,Z)-d^p_{\sigma,\beta}(\Gamma_{\tau,\alpha}\,Z)$.

Since $Z$ is adapted and $d^p_{\sigma,\beta}\Gamma_{\tau,\alpha}$ is only different from zero when $\tau<\sigma$, this last term does not contribute to the integration. We end up with

$$E\big(\delta(Z)\big)^2=E\left(\sum_{\alpha,\beta}\int_0^1\!\!\int_0^1 d^p_{\sigma,\beta}(\nabla_{\tau,\alpha}\,Z)d^p_{\tau,\alpha}(\nabla_{\sigma,\beta}\,Z)d\tau\,d\sigma\right)+I.$$

Finally, the fact that, for the 1-differential form associated to $Z$ we have

$$\begin{aligned}
\langle d\rho,\tilde e_{\tau,\alpha}\wedge\tilde e_{\sigma,\beta}\rangle=&\,-(\,\tilde e_{\tau,\alpha}\mid\nabla_{\sigma,\beta}\,Z\,)+(\,\tilde e_{\sigma,\beta}\mid\nabla_{\tau,\alpha}\,Z\,)\\
&+\big(\,T((\tau,\alpha),(\sigma,\beta))\;\big|\;Z\,\big),
\end{aligned}$$

allows to deduce the following result (cf. [6] for details):

**Theorem 7.3.** *There exists two operators on $H^1$, $A^0$ and $A^1$ such that, for any smooth adapted tangent vector field $Z$ we have*

$$E\big(\delta(Z)\big)^2+E\|d\rho_Z\|^2=E\|\nabla Z\|^2_{H^1\otimes H^1}+E\big(\,A^0(Z)\;\big|\;Z\,\big)+E\big(\,A^1(DZ)\;\big|\;Z\,\big),$$

*where $A^0$ and $A^1$ are given by $H^1$-operators with integral kernels defined in terms of stochastic integrals.*

This first result shows that, even for adapted vector fields, a Weitzenböck formula on the path space with respect to the Markovian connection gives rise to first order non trivial terms. At this stage we have considered the Markovian connection mainly because it naturally appears when dealing with first order commutation formulae, as seen in last paragraph. On the other hand derivating on the path space means derivating on the Wiener space with respect to tangent processes; as we have seen, assumptions on the *second* derivatives of $Z$ are needed to define $DZ$. One could of course expect things to be easier for Levi-Civita connection. In fact, this is far from being the case.

We first notice that, since we have an explicit control of the expressions $A^0$ and $A^1$ in terms of stochastic integrals, we can show that

$$E\big(\delta(Z)\big)^2 + E\|d\rho_Z\|^2 < +\infty$$

under suitable integrable assumptions on the vector field $Z$ (cf. [6]).

Now let us consider a Weitzenböck-type formula valid for such vector fields $Z$. It should be given by

$$E\big(\delta(Z)\big)^2 + E\|d\rho_Z\|^2 = E\|\tilde{\nabla} Z\|^2_{H^1\otimes H^1} + E\big(\,\tilde{\mathcal{R}}(Z) \mid Z\,\big). \tag{7.2}$$

The left-hand side of this equality being finite, let us look at $E\|\tilde{\nabla} Z\|^2_{H^1\otimes H^1}$ according to the expression obtained in Theorem 6.1 for the Levi-Civita covariant derivative. It is given in terms of a tangent process and therefore the $H^1$-norm will be infinite. We encounter here the problem already mentioned in paragraph 3 of the difficulty of defining a Riemannian metric for tangent processes.

More precisely we have:

**Theorem 7.4** (Explosion of the Levi-Civita Ricci tensor [6]). *The right-hand side of the identity* (7.2) *is a sum of two infinite terms even when the sum is finite.*

To prove this result we may take $\{\varphi_k\}$ an orthonormal basis of the space $H^1$ and write

$$E\|\tilde{\nabla} Z\|^2 = \sum_{k,l} \big(\,\tilde{\nabla}_{\varphi_l} Z \mid \varphi_k\,\big)_{H^1}.$$

Using Theorem 6.1, the first term corresponds to

$$\frac{1}{2} E \sum_{k,l} \left[\int_0^1 \Omega_{\alpha\beta\lambda}^{\ \ \gamma}\, \varphi_l^\alpha\, Z^\beta\, \dot{\varphi}_k^\gamma\, dx^\lambda\right]^2$$

and the energy identity for Itô integrals implies

$$\sum_k \big[\dot{\varphi}_k(\tau)\big]^2 = +\infty \quad \forall\tau.$$

7.4. **A modified Riemannian metric.** In [5] a Riemannian metric which takes into account the perturbation of the divergence due to the Ricci curvature term was considered. With respect to a connection defined accordingly, the first order commutation formula has a simplified expression.

We consider in $H$ the scalar product:

$$((\,h_1 \mid h_2\,)) = (\,\hat{h}_1 \mid \hat{h}_2\,)_{H^1},$$

where

$$\hat{h}(\tau) = h(\tau) + \frac{1}{2}\int_0^\tau Ric\,(h)\, ds,$$

and we define the covariant derivative of a constant tangent vector field on the path space $Z$ with respect to $h$ by

$$(\dot{\hat{\nabla}}_h Z)(\tau) = \int_0^\tau \Omega(\circ dx, h)\,\dot{\hat{z}}(\tau).$$

Then the following relation with the Markovian covariant derivative holds:

$$\hat{\nabla}_h Z = \nabla_h \hat{Z}.$$

The modified connection is still Riemannian and has a torsion.

**Theorem 7.5** (Commutation formula). *For $z, h \in H$, the following identity holds:*

$$D_h \delta(z) = \delta(\hat{\nabla}_h z) + ((\, z \mid h \,)).$$

We consider vector fields, which are of the form

$$Z(p)(\tau) = \sum_{k,\alpha} f_{k\alpha}(p)\, v_{k\alpha}(\tau)$$

where $f_{k\alpha}$ are cylindrical functions on $\mathbb{P}_{m_0}(M)$ and $v_{k\alpha}$ are the adapted vector fields defined by

$$\dot{v}_{k\alpha}(\tau) = \mathbf{1}_{\tau_k < \tau < \tau_k + 1}\, \varepsilon_\alpha$$

for a partition $\{\tau_k\}$ of the interval $[0,1]$.

Such processes were called by Fang (cf. [12]) *simple processes.*

The Weitzenböck formula corresponding to the covariant derivative $\hat{\nabla}$ allows to deduce the following estimation (cf. [5]):

**Theorem 7.6.** *There exists a constant $c > 0$ such that*

$$E(\delta Z)^2 \leq c(E\|Z\|^2 + E\|\hat{\nabla} Z\|^2)$$

*for every simple process $Z$.*

## 8. Anticipative integrals and Weitzenböck formulae

As we have discussed in section 4, in the Riemannian setting the notion of anticipative (Skorohod) integral no longer coincides, as is the case in $\mathbb{R}^d$, with the notion of divergence with respect to Wiener measure in a direct way. In fact, not only there is a correction term due to the Ricci tensor of the underlying manifold (already present in the adapted case) but an extra term involving the curvature appears (cf. formula (4.1)). In this section, when referring to anticipative stochastic integrals on the path space, we shall be talking in fact about divergences.

To obtain $L^P$-estimates for such divergences, it is enough to proceed by approximation by adapted vector fields and use the Weitzenböck formulae already developed for these fields.

For $q > 1$ we denote $\mathbb{D}_1^q$ the completion of the space of simple processes under the norm

$$\|Z\|^q_{\mathbb{D}_1^q} = E\left(\int_0^1 |Z(\tau)|^2\, d\tau\right)^{\frac{q}{2}} + E\left(\int_0^1\!\!\int_0^1 |D_\sigma Z(\tau)|^2\, d\tau, d\sigma\right)^{\frac{q}{2}}.$$

By an approximation procedure Fang showed in [12] that, if $Z$ belongs to a space $\mathbb{D}_1^q$ for some $q > 2$, then the divergence of $Z$ exists and

$$\|\delta(Z)\|_{L^2} \leq c_q \|Z\|_{\mathbb{D}_1^q}.$$

The same kind of approximation methods were used [6] with respect to the Markovian connection as well as in [5] with respect to the modified connection discussed in paragraph 7.4.

At this stage we could ask ourselves whether the passage from the adapted to the non adapted case is really a source of extra difficulties (with respect to the Wiener space situation). So far we have only looked at this passage from the point of view of estimating norms and not tried to obtain closed commutation formulae for anticipative vector fields.

The first order commutation relation for adapted fields has shown that

$$D_\sigma(\delta Z) = \delta(\nabla_\sigma Z) + B(Z),$$

where

$$\begin{aligned} B(Z) = d^p_\sigma Z - \frac{1}{2}\int_\sigma^1 Ric\,(\dot Z)\, d\tau + \frac{1}{4}\int_\sigma^1 Ric\,(Ric\,(Z))\, d\tau \\ + \frac{1}{2}(Ric\, Z)_\sigma + \frac{1}{2}\int_\sigma^1 \big[\nabla Ric\big](z)\cdot dx(\tau). \end{aligned}$$

Let $Z$ be an adapted tangent vector field and $f$ a smooth functional on the path space. We have

$$\begin{aligned} D_\sigma\big(\delta(fZ)\big) &= D_\sigma\big(f\delta Z - D_Z f\big) \\ &= (D_\sigma f)\,\delta Z + f\delta(\nabla_\sigma Z) + fB(Z)Z - D_\sigma D_Z f \end{aligned}$$

On the other hand,

$$\delta(\nabla_\sigma(fZ)) = f\delta(\nabla_\sigma Z) - D_{(\nabla_\sigma Z)} f + (D_\sigma f)\delta Z - D_Z D_\sigma f.$$

So, apart from the modification due to the Ricci tensor of the manifold, the difference between $D_\sigma(\delta(fZ))$ and $\delta(\nabla_\sigma(fZ))$ makes intervene the structure equations, which are, as we have seen, nontrivial on the path space.

We refer to [13] for developments of first order commutation formulae.

Let $\Delta_1$ denote the Laplacian on 1-forms associated to the Markovian connection, namely:

$$\Delta_1 Z = -\nabla^*\nabla Z.$$

We have

$$D_{\tau,\alpha}(\, Y \mid \nabla_{\tau,\alpha} Z\,) = (\,\nabla_{\tau,\alpha} Y \mid \nabla_{\tau,\alpha} Z\,) + (\, Y \mid \nabla_{\tau,\alpha}(\nabla_{\tau,\alpha} Z)\,)$$

and, since $E(\,\Delta_1 Z \mid Y\,) = -E(\,\nabla Z \mid \nabla Y\,)$, we derive the following expression, which holds for general (not necessarily adapted) vector fields $Z$:

$$\begin{aligned} d^p_{\sigma,\beta}(\Delta_1 Z) = \sum_\alpha \int_0^1 d^p_{\sigma,\beta}(\nabla^2_{\tau,\alpha} Z)\, d\tau - \int_0^1 d^p_{\sigma,\beta}(\nabla_{\tau,\alpha} Z)\circ dx^\alpha(\tau) \\ - \frac{1}{2}\int_0^1 Ric\,(\nabla_{\tau,\alpha} Z)_{(\sigma,\beta)}\circ dx^\alpha(\tau), \end{aligned} \tag{8.1}$$

where the Stratonovich integral is to be taken in the Stratonovich-Skorohod sense.

Let us denote by $\Box$ the de Rham-Hodge Laplacian, $\Box = d\delta + \delta d$, on forms of degree one.

**Theorem 8.1.** *There exists an operator on $H^1$, $A$, such that, for any smooth tangent vector field $Z$ we have*

$$(\Box + \Delta)Z_{(\sigma,\beta)} = A(Z)_{(\sigma,\beta)} +$$
$$\sum_{\alpha,\gamma} \int_0^1\int_0^1 D_{\rho,\gamma}\dot{z}_{\tau,\alpha}\left[d^p_{\sigma,\beta}T\big((\rho,\gamma),(\tau,\alpha)\big) - d^p_{\rho,\gamma}T\big((\tau,\alpha),(\sigma,\beta)\big)\right] d\tau\, d\rho,$$

*where $A$ has an integral kernel defined in terms of stochastic integrals.*

## 9. Adapted differential geometry

We are interested in considering Weitzenböck formulae for exact differential forms $\omega = df$. If $\mathcal{L}$ denotes the Ornstein-Uhlenbeck operator on the path space, defined by

$$\mathcal{L}f = -\delta D f, \tag{9.1}$$

this means computing the commutator between $d\mathcal{L}f$ and $\Delta_1(df)$.

In this section (and following [8]) we consider a type of renormalization that consists in restricting identities (such as Weitzenböck formulae) to adapted vector fields. We refer also to [4] where, in the same spirit, a modified Markovian connection has been defined.

In a properly defined adapted differential geometry many identities simplify drastically. The main result is that, through this renormalization by restriction the Ricci tensor associated to the Markovian connection on the path space is equal to the identity. In adapted differential geometry 1-differential forms are not identified via the Hilbertian structure with vector fields; this allows to consider simultaneously closed forms and adapted vector fields in duality.

Let us consider the family of projectors on $H^1 = H^1([0,1];\mathbb{R}^d)$, $\Pi_\sigma$, for $\lambda \in [0,1]$, defined by

$$d_\tau(\Pi_\lambda z) = \mathbf{1}_{\tau<\lambda}\dot{z}(\tau).$$

This family corresponds to the Itô time filtration.

We consider $\mathfrak{A}$, the group of unitary transformations of $H^1$ that commute with the projectors $\Pi_\sigma$, thus restricting the group of all unitary transformations which would a priori define the "orthonormal frames" on the path space.

Denoting, respectively, $GL(d)$ and $O(d)$ the linear group and the orthogonal group of $\mathbb{R}^d$, and $\mathbb{P}(*)$ the bounded measurable maps of $[0,1]$ into $*$, we can identify $\mathbb{P}(O(d))$ and $\mathbb{P}(GL(d))$ to, respectively $\mathfrak{A}$ and to the group of bounded linear transformations of $H^1$ commuting with the family $\Pi_\sigma$, through the following action:

$$(u * z)(\tau) = \int_0^\tau u(\sigma)\,\dot{z}(\sigma)\,d\sigma.$$

The Lie algebra of the group $\mathfrak{A}$ can be identified with $\mathbb{P}(\mathrm{so}(d))$.

**Definition 9.1.** We call *frame* at a point $p \in \mathbb{P}_{m_0}(M)$ an isometric surjective map of $H^1$ into the space of tangent vector fields on the path space.

We call *adapted frame* a frame which intertwines with the family of projection operators on the space of tangent vector fields defined by:

$$(\Pi_\lambda(Z))(\tau) = Z_\tau \quad \forall\tau < \lambda,$$
$$(\Pi_\lambda(Z))(\tau) = t^p_{\tau\leftarrow\lambda}(Z_\lambda) \quad \forall\tau > \lambda.$$

The frame bundle $O(\mathbb{P}_{m_0}(M)$ will consist of the collection of all adapted frames.

Using the notation $\Theta$ already used for the canonic frame defined by the parallel transport, $\Theta(Z)_\tau = t^p_{0\leftarrow\tau} Z_\tau = z_\tau$, the map $z \overset{\chi}{\mapsto} \Theta_p^{-1}(u * z)$ defined for $u \in h$, $z \in H^1$, is a bijective isomorphism from $\mathfrak{A} \times \mathbb{P}_{m_0}(M)$ to $O(\mathbb{P}_{m_0}(M))$.

Let $\sigma(p) = \chi(e,p)$, where $e$ denotes the constant path equal to the identity; $\sigma$ is the section of the bundle of adapted frames. If $\varphi : O(\mathbb{P}_{m_0}(M)) \to \mathbb{P}_{m_0}(M)$ denotes the canonic projection, then $\varphi \circ \sigma = \mathrm{Id}_{\mathbb{P}_{m_0}(M)}$.

For fixed $u \in \mathfrak{A}$, the map

$$
\begin{aligned}
Q_u &: O(\mathbb{P}_{m_0}(M)) \to O(\mathbb{P}_{m_0}(M)) \\
Q_u(r) &= r \circ u^{-1}
\end{aligned}
$$

defines a group action of $\mathfrak{A}$ on $O(\mathbb{P}_{m_0}(M))$.

The Markovian connection we have defined in section 6, namely

$$
\Gamma_{z,p}(\tau) = \int_0^\tau \Omega(z(\sigma), \circ dp(\sigma)),
$$

has a martingale part belonging to $\mathbb{P}(\mathrm{so}(d))$. An important observation is that, when the underlying manifold has a zero Ricci curvature, then $\Gamma_{z,p} \in \mathbb{P}(\mathrm{so}(d))$ since, by Bianchi identities, the contraction in the stochastic integral disappears.

The Markovian connection defines a family of canonic horizontal vector fields on the frame bundle $O(\mathbb{P}_{m_0}(M))$, $A_Z$. We define $A_Z(r)$ at a point $r = \sigma(p) \circ u$ by

$$
A_Z(r) = (\tilde{\Gamma}_Z(r), \tilde{Z}(r)), \tag{9.2}
$$

where

$$
\begin{aligned}
&\tilde{\Gamma}_Z(\sigma(p)\, u) = u^{-1} \circ \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \exp(-\varepsilon \Gamma_{u*z,p}) \circ u, \\
&\tilde{Z}(r) = r(z).
\end{aligned}
$$

Given a tangent vector field $Z$ on the path space $F_Z(r) = r^{-1}(Z_{\varphi(r)})$ defines its *scalarization*. The Markovian covariant derivative $\nabla_Z Y$ is expressed on the frame bundle by

$$
F_{\nabla_{r(z)} Y}(r) = \langle dF_Y, A_z \rangle_r \,. \tag{9.3}
$$

We may then consider $\tilde{D}_z$, the directional derivative along the vector field $A_z$, operating on smooth cylindrical functionals on $O(\mathbb{P}_{m_0}(M))$. The covariant derivative $\nabla_{\tau,\alpha}$ is defined by

$$
(\nabla_{\tau,\alpha}\Phi)_p = (\tilde{D}_{\tau,\alpha}\Phi)_{\sigma(p)} \tag{9.4}
$$

for a "vector field" $\Phi : O(M) \to L^2([0;1]; \mathbb{R}^d)$.

On the frame bundle $O(\mathbb{P}_{m_0}(M))$ we can define a parallelism, by considering a 1-differential form with values in $H^1 \times \mathfrak{A}$. Let $\pi = (\pi_1, \pi_2)$ denote this differential from, defined by

$$
\begin{aligned}
\langle \pi_1, T\rangle_r &= r^{-1}(\varphi'(r)(T)), \\
\langle \pi_2, T\rangle_r &= \tilde{\Gamma}_{\dot{\pi}(T)} + \psi'(r)\, T,
\end{aligned} \tag{9.5}
$$

where $\psi$ denotes the projection of the domain of the map $\chi$ on the first component and where the tangent space at the point $u \in \mathfrak{A}$ is identified to

$$
\left\{\, u \exp^{\varepsilon g} : g \in \mathbb{P}(\mathrm{so}(d)) \,\right\}.
$$

**Theorem 9.2** ([8]). *The structural equations of the frame are*

$$\big(\langle d\pi_1, T_1 \wedge T_2\rangle + \pi_2(T_1)\,\pi_1(T_2) - \pi_2(T_2)\,\pi_1(T_1)\big)_{\sigma(p)} = \\ -\int_0^{\cdot} \Omega\big(\pi_1(T_2), \pi_1(T_1)\big) \circ dx$$

$$\big(\langle d\pi_2, T_1 \wedge T_2\rangle - \pi_2(T_1)\,\pi_2(T_2) + \pi_2(T_2)\,\pi_2(T_1)\big)_{\sigma(p)} = \\ -[\Gamma_{z_1}, \Gamma_{z_2}] - (D_{z_1}\Gamma_{z_2}) + (D_{z_2}\Gamma_{z_1}) + \Gamma_{[z_1,z_2]},$$

*where* $z_i = \pi_1(T_i)$, $i = 1, 2$.

From the expressions of the last theorem we recover the formula for the torsion of the Markovian connection, namely

$$\mathcal{T}(z_1, z_2) = -\int_0^{\cdot} \Omega(z_1, z_2) \circ dx,$$

and we obtain the curvature tensor, which is equal to:

$$\mathcal{C}(z_1, z_2) = -[\Gamma_{z_1}, \Gamma_{z_2}] - (D_{z_1}\Gamma_{z_2}) + (D_{z_2}\Gamma_{z_1}) + \Gamma_{[z_1,z_2]}.$$

**Corollary.** *The Ricci type trace of the curvature of the Markovian connection, namely*

$$\mathrm{Trace}\,\mathcal{C}(z) = \sum_\alpha \int_0^1 \mathcal{C}(z, \tilde{e}^\alpha_\tau) * e^\alpha_\tau \, d\tau$$

*is given by*

$$d^p_\tau \,\mathrm{Trace}\,\mathcal{C}(z) = Ric_{p(\tau)}\,(d^p_\tau Z).$$

In particular, if $\mathrm{Ricci}(M) = 0$, then the Ricci trace on the path space vanishes.

We notice that these results are a consequence of the Markovian character of the covariant derivative on the path space (cf. [8]).

As we have already pointed out, when the manifold is Ricci flat we can replace the Stratonovich integral defining the Markovian connection by an Itô integral, since the contraction term vanishes; we have an analogous situation concerning the torsion. On the other hand,

**Theorem 9.3.** *For* $Z_i$, $i = 1, 2$, *two adapted tangent vector fields, if*

$$\mathcal{T}(Z_1, Z_2)(\tau) = -\int_0^\tau \Omega(Z_1, Z_2)\, dx$$

*then* $E_\mu(D_{\mathcal{T}} F) = 0$ *for every smooth functional* $F$.

The Markovian character of the connection together with the simplification in the stochastic integrals when $M$ is Ricci flat (as in theorem 9.3 above) induce drastic simplifications on the corresponding Weitzenböck formula in this adapted differential geometry.

In the situation where $\mathrm{Ricci}(M) = 0$ (which does not imply that the curvature tensor of $M$ is trivial), the expression for the Ornstein-Uhlenbeck operator on the path space is (cf. [16]):

$$\mathcal{L}f = \frac{1}{2}\sum_\alpha \int_0^1 D^2_{\tau,\alpha}\, d\tau - D_{\tau,\alpha} \circ dx^\alpha(\tau). \tag{9.6}$$

**Theorem 9.4** ([8]). *Let* $\mathrm{Ricci}(M) = 0$. *If* $\Delta_1$ *denotes the Laplacian on vector fields, namely*

$$\Delta_1 = \frac{1}{2}\sum_{\alpha}\int_0^1 \nabla^2_{\tau,\alpha}\,d\tau - \int_0^1 \nabla_{\tau,\alpha}\circ dx^{\alpha}(\tau),$$

*then* $\Delta_1 Z$ *is an adapted vector field for every adapted tangent vector field* $Z$ *and, for every smooth functional* $f$, *the following identity holds:*

$$E\left\langle d\mathcal{L}f, Z\right\rangle + E\left\langle df, Z\right\rangle = E\left\langle df, \Delta_1 Z\right\rangle.$$

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