PERIOD AND ENERGY IN ONE DEGREE OF FREEDOM SYSTEMS

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Abstract. For one degree of freedom systems there exists a very well known formula for the period (T) of periodic solutions. In this note we give a detailed description of the behavior of T in function of the total energy (E), near a stable equilibrium position of energy E_0. A formula for $\frac{dT}{dE}(E_0)$ is established. We illustrate these formulae with two examples. The second one is a new proof of a Bertrand's theorem.

1. Introduction

As it is very well known, a system with one degree of freedom is a differential equation of the form

$$m\ddot{x} = -U'(x), \tag{1}$$

where $m > 0$, U, the potential, is a smooth function of x, defined in a real interval, and $\ddot{x} \equiv \frac{d^2x}{dt^2}$, $U' \equiv \frac{dU}{dx}$. Equation (1) is nothing more than a Newton's equation of motion in one dimension. We follow essentially [1].

If $t \mapsto x(t)$ is a solution of (1), then

$$E \equiv E(x(t), \dot{x}(t)) = \frac{m}{2}\dot{x}(t)^2 + U(x(t)),$$

the energy, is constant.

Let ξ be such that $U'(\xi) = 0$. Then, ξ is called an equilibrium position; $t \mapsto x(t) \equiv \xi$ is a solution of (1). We shall assume that $U''(\xi) > 0$, which implies that ξ is a stable equilibrium position. Denote $E_0 = E(\xi, 0)$.

Under the above assumption, for E near and $> E_0$, there are periodic movements $t \mapsto x(t)$ around ξ and of energy E. The period T of such movements

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is given by

$$T(E) = 2\int_{x_1}^{x_2}\left[\frac{2}{m}(E - U(x))\right]^{-\frac{1}{2}} dx, \tag{2}$$

where $x_i \equiv x_i(E)$ are the turning points, $U(x_i(E)) = E$, $i = 1, 2$.

In the limit $E \longrightarrow E_0$ one obtains the well known formula

$$T(E_0) \equiv \lim_{E\to E_0} T(E) = 2\pi\left[\frac{1}{m}U''(\xi)\right]^{-\frac{1}{2}}.$$

The object of this note is to study the behavior of $T(E)$ around E_0 (formula (3)) and compute $\frac{dT}{dE}(E_0)$ (formula (4)).

In order to illustrate the usefulness of this formula we give two examples. The second one is another proof of a well known result on the types of central forces that lead to periodic movements only.

2. The behavior of $T(E)$ near an equilibrium position

Assume that U is of class C^n, $n \geq 2$, and that $U'' > 0$ in $[x_1, x_2]$. Define the function $x \mapsto X = \varphi(x)$, for $x \in [x_1, x_2]$, such that

$$\varphi(x) = \pm\sqrt{\frac{2(U(x) - E_0)}{U''(\xi)}},$$

where $+$ stands for $x \geq \xi$ and $-$ stands for $x \leq \xi$; φ is of class C^{n-1} and $\varphi'(x) = U''(\xi_2)(U''(\xi)U''(\xi_1))^{-\frac{1}{2}} > 0$, with ξ_1 and ξ_2 between ξ and x. Denote $[-X_0, X_0] = \varphi([x_1, x_2])$ and consider the function $f : [-X_0, X_0] \to \mathbb{R}$, such that

$$f(X) = \frac{1}{\varphi'(\varphi^{-1}(X))}.$$

The function f is of class C^{n-2}. Denote $f'(X) = \frac{df(X)}{dX}$, $f''(X) = \frac{d^2 f(X)}{dX^2}$, and so on. For $n \geq 4$, and as $U''(\xi)X = U'\left(\varphi^{-1}(X)\right)f(X)$, one has

$$f(0) = 1,\ f'(0) = -\frac{U'''(\xi)}{3U''(\xi)},\ f''(0) = \frac{5U'''(\xi)^2 - 3U^{(4)}(\xi)U''(\xi)}{12U''(\xi)^2}.$$

Formula (2) can be written

$$T(E) = 2\int_{-X_0}^{X_0} f(X)\left[\frac{1}{m}\left(2(E - E_0) - U''(\xi)X^2\right)\right]^{-\frac{1}{2}} dX.$$

Writing $f(X) = 1 + f'(0)X + \frac{1}{2}f''(\theta)X^2$, for some $\theta \in]-X_0, X_0[$, and as

$$\int_{-X_0}^{X_0} X^2\left[\frac{1}{m}\left(2(E - E_0) - U''(\xi)X^2\right)\right]^{-\frac{1}{2}} dX = \frac{\pi\sqrt{m}}{U''(\xi)^{\frac{3}{2}}}(E - E_0),$$

one has

$$T(E) - T(E_0) = \sigma \frac{\pi\sqrt{m}}{U''(\xi)^{\frac{3}{2}}}(E - E_0), \tag{3}$$

for some $\sigma \in f''([-X_0, X_0])$.

Hence

(4)

$$T'(E_0) \equiv \frac{dT}{dE}(E_0) = f''(0)\frac{\pi\sqrt{m}}{U''(\xi)^{\frac{3}{2}}} = \pi\sqrt{m}\left(\frac{5U'''(\xi)^2 - 3U^{(4)}(\xi)U''(\xi)}{12U''(\xi)^{\frac{7}{2}}}\right).$$

Example 1. *Consider a mathematical pendulum of length l under the free fall acceleration g. The potential is $U(x) = -\frac{g}{l}\cos x$, where x is the angle of deviation of the pendulum from the vertical.*

In this case $\varphi(x) = 2\sin\frac{x}{2}$ and $f(X) = 2\left(4 - X^2\right)^{-\frac{1}{2}}$. Simple calculations show that formula (3) becomes

$$T(E) - T(E_0) = 2\sigma\pi\sqrt{\frac{l}{g}}\left(\sin\frac{x_0}{2}\right)^2, \text{ for some } \sigma \in \left[\frac{1}{4}, \frac{1 + 2\left(\sin\frac{x_0}{2}\right)^2}{4\left(\cos\frac{x_0}{2}\right)^5}\right],$$

where x_0 is the maximum deviation angle.

Example 2. *The following example is another proof of a Bertrand's theorem. We follow partially [1], Chapter 2, § 8.D. See also [2], page 51 and [4], page 90. The original proof is in [3].*

Let U be the potential of a central force, μ the angular momentum and V the effective potential, $V(r) = U(r) + \frac{\mu^2}{2mr^2}$, where $t \mapsto r(t)$ is the radial movement. Assume that there are periodic radial movements and that r_0 is a minimum for V. Then $t \mapsto r_0$ is a stable circular orbit. Let $t \mapsto r(t)$ be a periodic radial movement, Φ the angle between a pericenter and an apocenter which are adjacent and let r_1 and r_2 be the distances from the pericenters and the apocenters to the center of the field, $r_1 < r_0 < r_2$. Making the change $x = \frac{1}{r}$, $x_1 = \frac{1}{r_1}$, $x_2 = \frac{1}{r_2}$, $x_0 = \frac{1}{r_0}$, $W(x) = U\left(\frac{1}{x}\right) + \frac{\mu^2}{2m}x^2$, one has $\Phi = \frac{\mu}{2\sqrt{m}}T(E)$, where $T(E)$ is as in (2) with U replaced by W and m by 1. As in [1] one easily obtains

$$\lim_{r_1,r_2\to r_0}\Phi = \Phi_{circ} = \lim_{E\to E_0}\frac{\mu}{2\sqrt{m}}T(E)$$
$$= \pi\mu\left[mW''(x_0)\right]^{-\frac{1}{2}} = \pi\left(\frac{U'(r_0)}{3U'(r_0) + r_0U''(r_0)}\right)^{\frac{1}{2}}.$$

As in [1] we consider the differential equation $U'(r) = C(3U'(r) + rU''(r))$, for r in some interval and $C > 0$. The solutions are $U(r) = ar^{\varepsilon}$ and $U(r) =$

$b\log r$, with $\varepsilon, a, b \neq 0$ and $\varepsilon > -2$. The second case ($b\log r$) can easily be excluded.

As Φ does not depend on E, $T(E)$ does not depend on E. Hence $\frac{dT}{dE}(E_0) = 0$. From (4), one has

$$5W'''(x_0)^2 - 3W^{(4)}(x_0)W''(x_0) = 0. \tag{5}$$

As $W(x) = ax^{-\varepsilon} + \frac{\mu^2}{2m}x^2$, we have that $x_0 = \left(\frac{\varepsilon am}{\mu^2}\right)^{\frac{1}{\varepsilon+2}}$ which, together with (5), implies $\varepsilon = -1$ or $\varepsilon = 2$. Remember that, in order to have bounded movements, for $\varepsilon = -1$, $a < 0$, and for $\varepsilon = 2$, $a > 0$.

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