POLYHEDRON PUZZLES AND GROUPS

JORGE REZENDE

1. Introduction

Consider a polyhedron. For example, a platonic, an archimedean, or a dual of an archimedean polyhedron. Construct flat polygonal plates in the same number, shape and size as the faces of the referred polyhedron. Adjacent to each side of each plate draw a number like it is shown in figures 1-10. Some of the plates, or all, can have numbers on both faces. We call these plates, two-faced plates. In this article, the two-faced plates have the same number adjacent to the same side. Figure 4 shows twenty faces of triangular two-faced plates and their reverse faces are represented in figure 5. Figure 7 shows twelve faces of pentagonal two-faced plates and their reverse faces are represented in figure 8.

Now the game is to put the plates over the polyhedron faces in such a way that the two numbers near each polyhedron edge are equal. If there is at least one solution for this puzzle one says that we have a polyhedron puzzle with numbers.

In this article we begin in Section 2 by giving some examples of puzzles. In Section 3 we describe the tetrahedron, the octahedron and the icosahedron symmetry groups. In Section 4 we recall some definitions on puzzle solutions and their relation with permutation groups. In Section 5 we show how to define puzzles using group theory.

This is the first of a series of two articles. In the next one [8], we suggest some simple mathematical activities using polyhedron puzzles.

2. Definitions using combinatorics

2.1. **Platonic and archimedean polyhedra.** From now on, assume that the numbers belong to the set $\{1, 2, \ldots, n\}$, and that all the numbers are used.

If we have plate faces which have the shape of a regular polygon with j sides, one can ask how many possible ways ν are there to draw the

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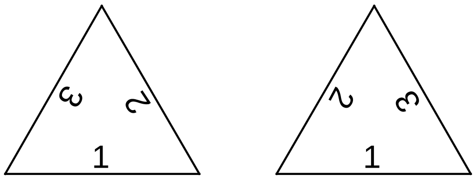

FIGURE 1.

numbers $1, 2, \ldots, n$, without repeating them on each plate face. The answer is

a) For $j = 3$ (equilateral triangle) and $n = 3$, then $\nu = 2$ (see figure 1).

b) For $j = 3$ and $n = 4$, then $\nu = 8$ (see figure 2); 8 **is precisely the number of the octahedron faces**. With these 8 plates we make the octahedron puzzle ($\equiv$ the octahedron (1) puzzle).

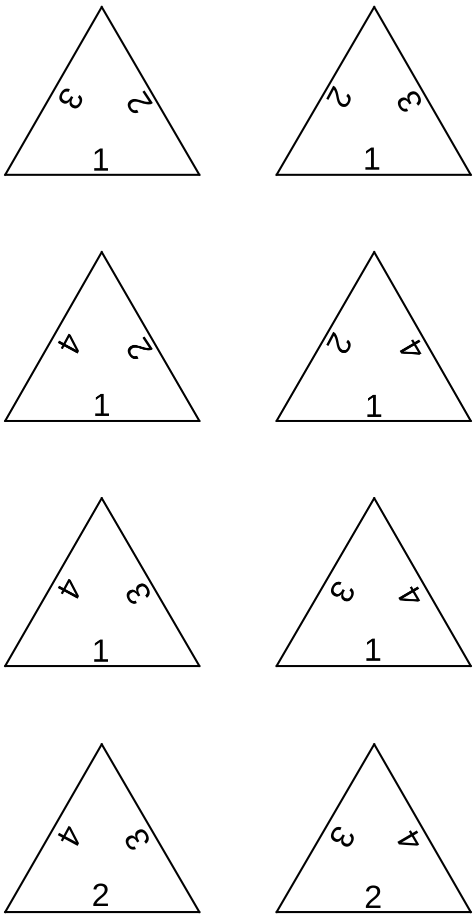

FIGURE 2.

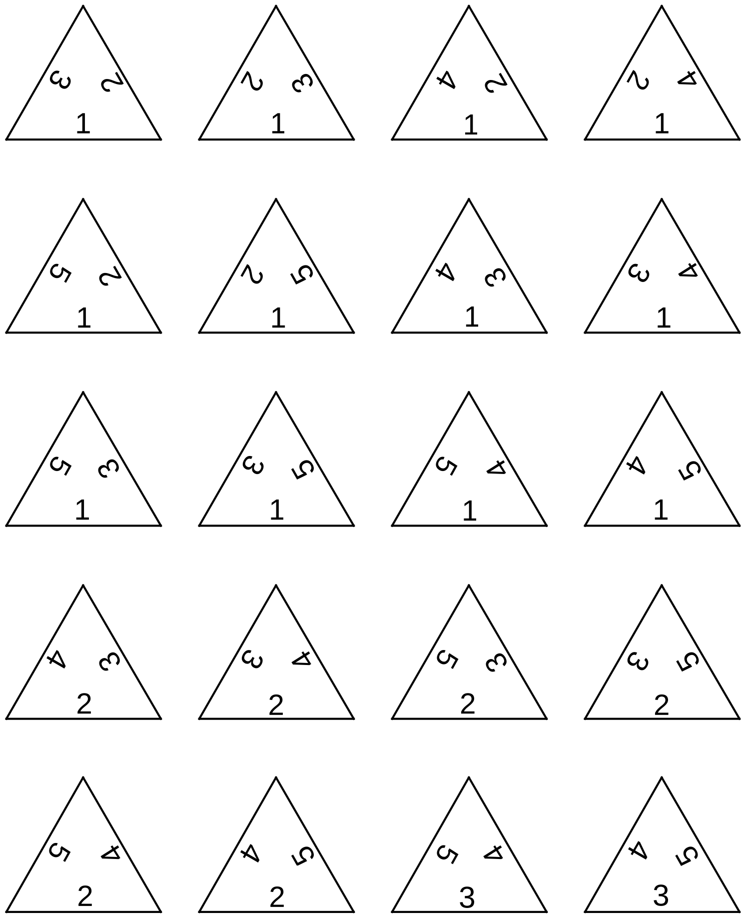

FIGURE 3.

c) For $j = 3$ and $n = 5$, then $\nu = 20$ (see figure 3); 20 **is precisely the number of the icosahedron faces**. With these 20 plates we make the icosahedron (1) puzzle.

d) For $j = 3$ and $n = 6$, then $\nu = 40$ (see figures 4 and 5); 40 **is precisely the double of the number of the icosahedron faces**. Construct different plates with the numbers written on both faces. This gives 20 plates. We call the related puzzle, the icosahedron second puzzle (or icosahedron (2)).

e) Consider again $j = 3$ and $n = 6$. Construct different plates with the numbers written only on one face, but in such a way that the numbers grow if we read them, beginning with the minimum, counter clock-wise. This gives 20 plates (see figure 4). We call the related puzzle, the icosahedron third puzzle (or icosahedron (3)).

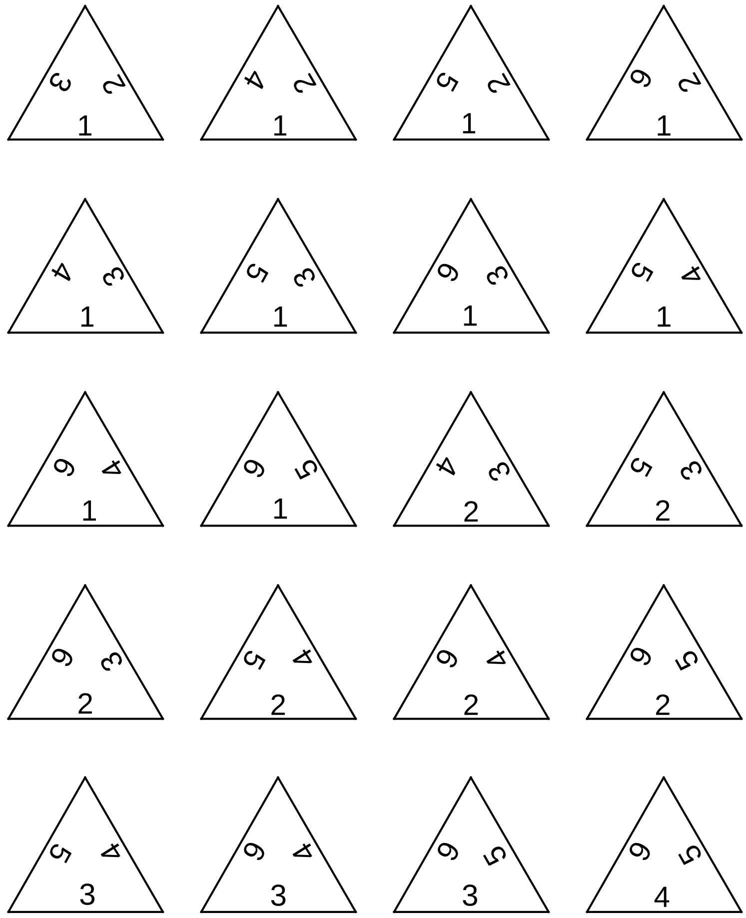

FIGURE 4.

f) For $j = 4$ (square) and $n = 4$, then $\nu = 6$ (see figure 6); 6 **is precisely the number of the cube faces**. With these 6 plates we make the cube puzzle ($\equiv$ the cube (1) puzzle).

g) For $j = 5$ (regular pentagon) and $n = 5$, then $\nu = 24$ (see figures 7 and 8); 24 **is precisely the double of the number of the dodecahedron faces**.

Construct different plates with the numbers written on both faces. This gives 12 plates. We call the related puzzle, the dodecahedron first puzzle (or dodecahedron (1)).

h) Let again $j = 5$ and $n = 5$. Construct different plates with the numbers written only on one face, but in such a way that the numbers read counter clock-wise, $abcd5$, are such that $abcd$ form an even permutation. This gives 12 plates (see figure 9). We call the related puzzle, the dodecahedron second puzzle (or dodecahedron (2)).

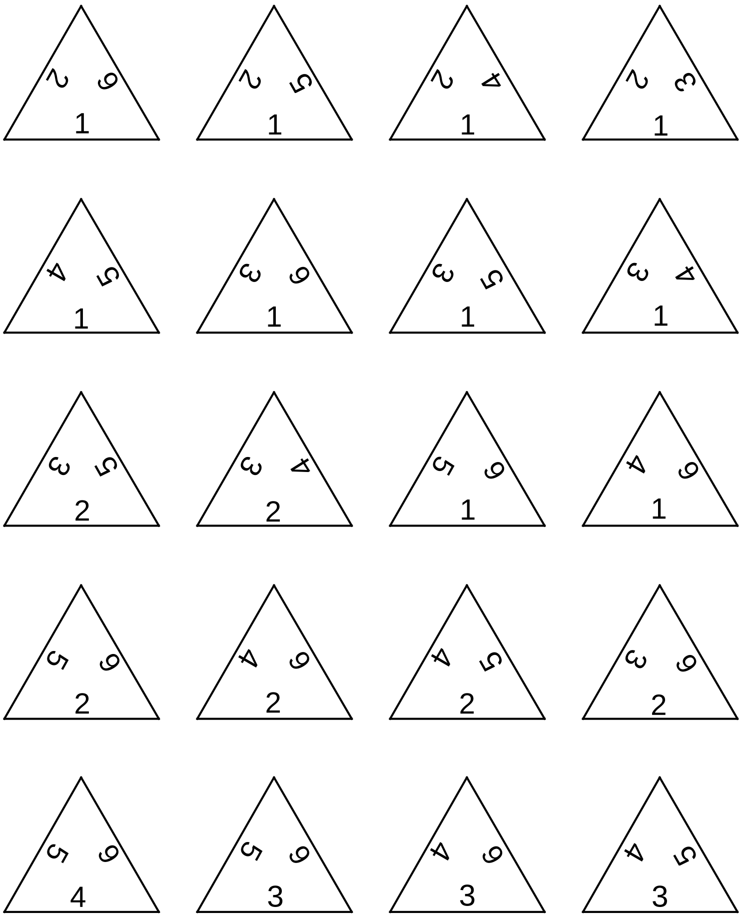

FIGURE 5.

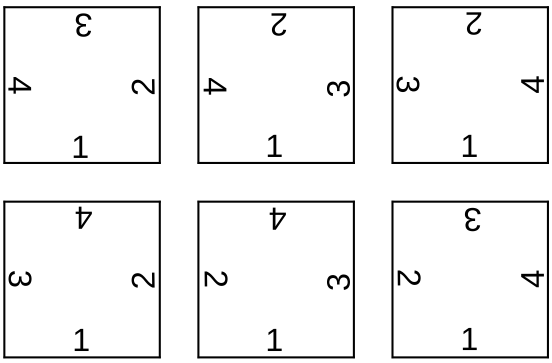

FIGURE 6.

i) Consider $n = 4$. With $j = 3$, one has $\nu = 8$ (see figure 2). With $j = 4$, one has $\nu = 6$ (see figure 6). Notice that 8 is precisely the number

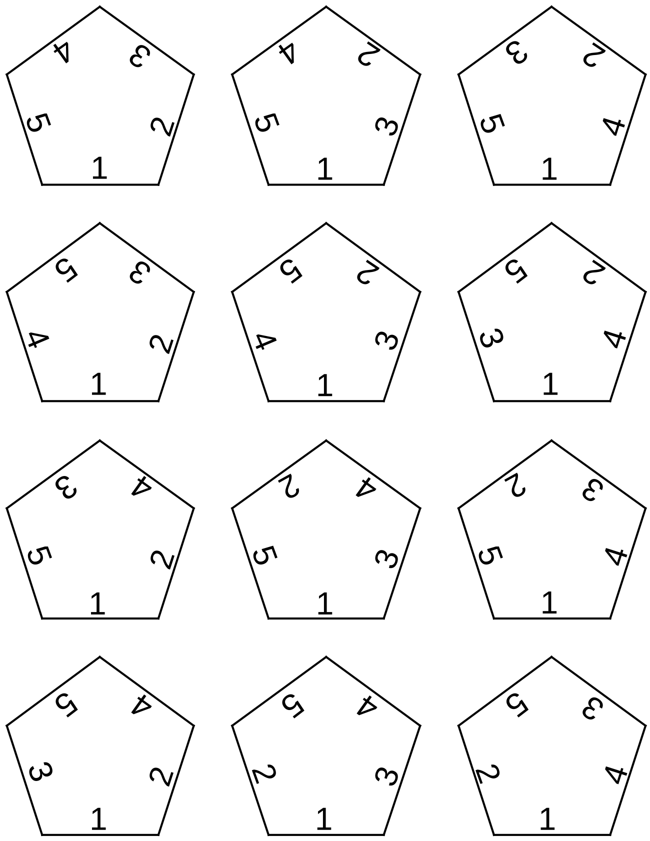

FIGURE 7.

of the cuboctahedron triangular faces and 6 is precisely the number of its square faces. This is an example of an interesting puzzle using an archimedean polyhedron. We call it the cuboctahedron puzzle (≡ the cuboctahedron (1) puzzle).

The general formula for ν is

$$\nu = (j-1)!\binom{n}{j} = \frac{n!}{(n-j)!j}.$$

2.2. **More puzzles.** Take now a deltoidal icositetrahedron. It has 24 deltoidal faces. If we have 24 plates which have the deltoidal shape the number of possible different ways to draw the numbers $1, 2, 3, 4$, without repeating them on each plate is precisely 24. This an example of an interesting puzzle using a dual of an archimedean polyhedron (see reference [6]).

Consider again the cube. It has 6 faces that are squares. The number of possible different ways to draw the numbers $1, 2, 3, 4$, with two repetitions

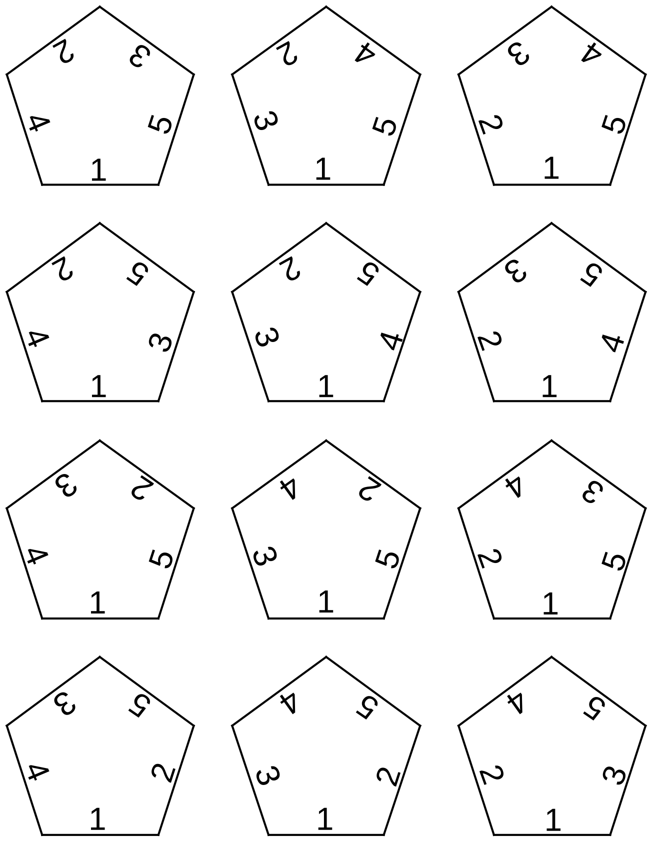

FIGURE 8.

of the form *aabb* (the numbers are read counter clock-wise) on each square plate is precisely 6. This gives the cube (2) puzzle.

Consider again the icosahedron. It has 20 faces that are equilateral triangles. The number of possible different ways to draw the numbers $1, 2, 3, 4, 5$, with one repetition on each triangular plate is precisely 20. This gives the icosahedron (4) puzzle (see reference [7]).

These are simple examples of polyhedron puzzles with numbers, which are enough in order to understand the following sections. There are, obviously, others as we shall see. For more examples see reference [2], which is a development of reference [1]. Reference [3] is a collection of some of these puzzles paper models.

3. Polyhedron symmetries

Consider a polyhedron in $\mathbb{R}^3$. From now on V denotes the set of the polyhedron vertices, E denotes the set of the polyhedron edges and F denotes the set of the polyhedron faces.

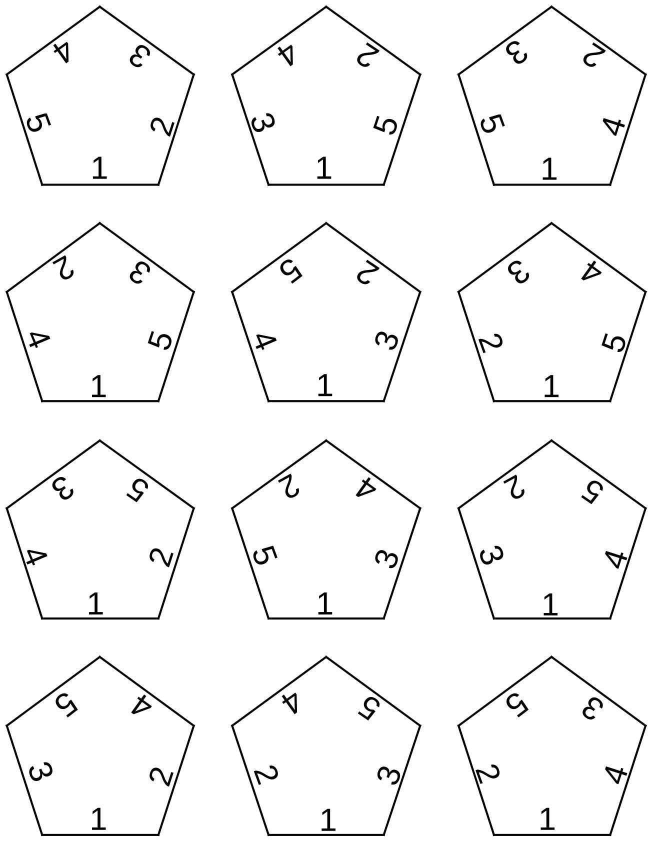

FIGURE 9.

The group of the polyhedron symmetries, Ω, called the polyhedron group, is the set of all isometries ω of $\mathbb{R}^3$, that send vertices to vertices, which implies that they send edges to edges, faces to faces. Every symmetry $\omega \in \Omega$ induces three bijections, that we shall also denote by ω, whenever there is no confusion possible: $\omega : V \to V$, $\omega : E \to E$ and $\omega : F \to F$. Denote also by $\Omega \equiv \{\omega : V \to V\} \equiv \{\omega : E \to E\} \equiv \{\omega : F \to F\}$, the three sets of these functions. One can say that each one of these three sets Ω is the set of the polyhedron symmetries. Notice that not all one-to-one functions $F \longrightarrow F$, $E \longrightarrow E$, $V \longrightarrow V$ are in Ω. With the composition of functions each one of these three sets Ω forms a group that is isomorphic to the group of the polyhedron symmetries. If $\omega_1, \omega_2 \in \Omega$, we shall denote $\omega_1\omega_2 \equiv \omega_1 \circ \omega_2$.

When no confusion is possible, $\omega \in \Omega$ represents also the group isomorphism $\omega : \Omega \to \Omega$, $\omega(\omega_1) = \omega\,\omega_1\omega^{-1}$, for every $\omega_1 \in \Omega$. Note that ω_1 and $\omega(\omega_1)$ have the same order. Look at the octahedron in figure 18. If ω is a counter clock-wise rotation of 90° around the z-axis, and ω_1 is

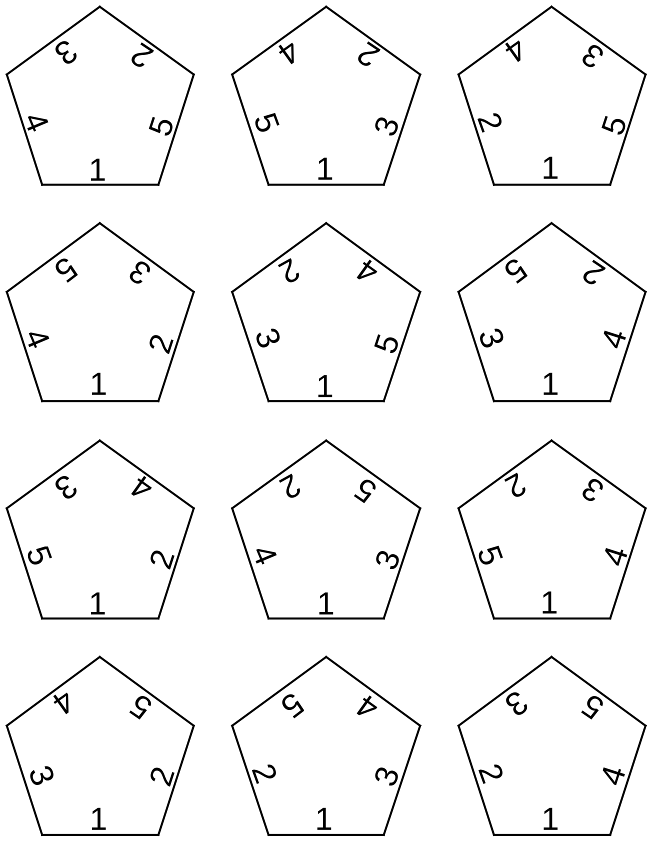

FIGURE 10.

a counter clock-wise rotation of 90° around the x-axis, then $\omega(\omega_1)$ is a counter clock-wise rotation of 90° around the y-axis. In simple words, ω transports x over y. Here, a counter clock-wise rotation around the z-axis, for example, means that we look from the positive z-semiaxis.

If Ω_1 is a subgroup of Ω, then Ω_1 acts naturally on the face set, F: for $\omega \in \Omega_1$ and $\varphi \in F$, one defines the action $\omega\,\varphi = \omega(\varphi)$.

In the following we only consider polyhedra centered at the origin, Ω^+ denotes the subgroup of Ω of the symmetries with determinant 1 and Ω^- denotes the subgroup of Ω of the symmetries with determinant -1.

3.1. **The tetrahedron group.** Consider the tetrahedron (see figures 11 and 12) and its group, Ω.

An element of Ω is, for example, the function $\omega(x, y, z) = (-x, y, z)$, that induces the function $\omega : E \longrightarrow E$

$$\omega(e_1) = e_1 \qquad \omega(e_2) = e_3 \qquad \omega(e_3) = e_2$$
$$\omega(e_4) = e_5 \qquad \omega(e_5) = e_4 \qquad \omega(e_6) = e_6.$$

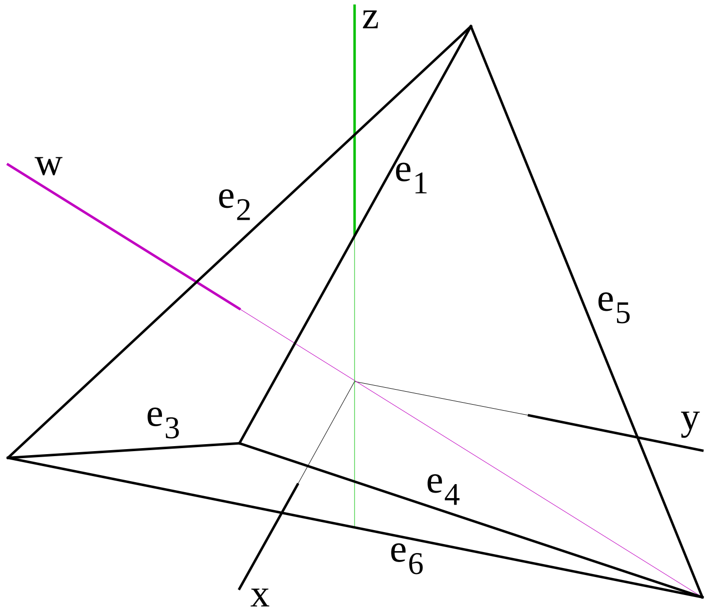

Figure 11.

This symmetry has determinant -1 and is one of the twelve elements of Ω with determinant -1. They reverse the orientation. The tetrahedron has no central symmetry.

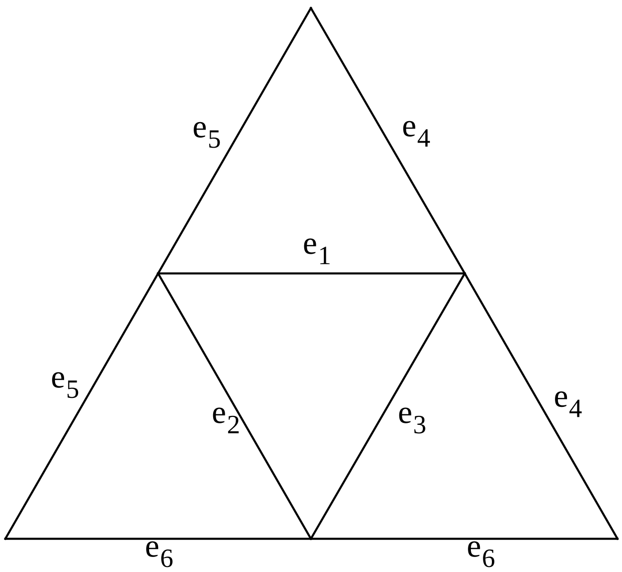

Figure 12.

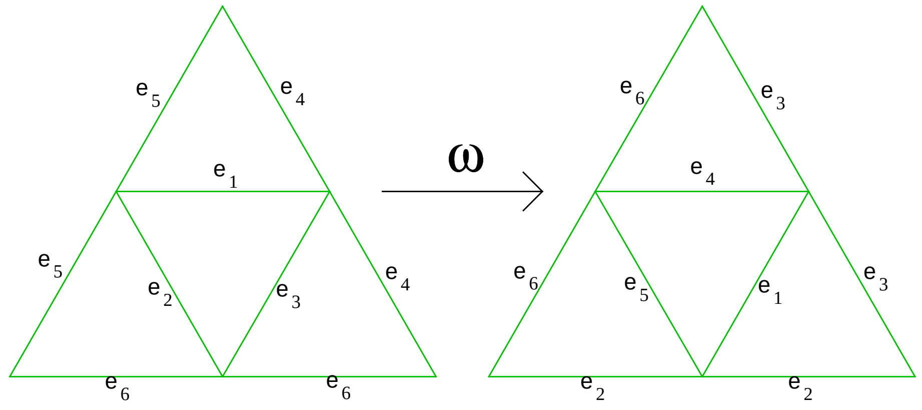

FIGURE 13.

The symmetries with determinant 1 (Ω^+), can be seen like this: one transports a chosen face in such a way that it goes to one of the four tetrahedron faces; as one has three possibilities of making them coincide (they are equilateral triangles), there are 12 (3×4) symmetries with determinant 1. Figure 13 shows one of these symmetries. In this case the function $\omega : E \to E$, is the following:

$$\begin{array}{lll} \omega(e_1) = e_4 & \omega(e_2) = e_5 & \omega(e_3) = e_1 \\ \omega(e_4) = e_3 & \omega(e_5) = e_6 & \omega(e_6) = e_2. \end{array}$$

The advantage of describing in this way the symmetries of Ω^+ is that it can be easily adapted to other polyhedra, and used in their computation in a computer program.

Another way of counting the symmetries de Ω^+ is the following: the identity (1); the rotations of 180° around the three axes defined by the centers of opposite edges (3); the rotations of 120° and 240° around the four axes defined by each vertex and the center of the opposite face (8).

The symmetries with determinant -1 (Ω^-) are the compositions of the symmetries de Ω^+ with a symmetry with determinant -1. The cardinal of Ω, the order of Ω, is, therefore, 24.

Let us see two examples:

a) Consider, in the figures 11 and 14, the rotations of 0° and 180° around the z-axis. They form a subgroup of Ω^+ of order two. The action of these rotations on the tetrahedron faces, sends every green face to the place of the other green face and every blue face to the place of the other blue face. The set of the green faces and the set of the blue faces are the orbits of this action (see figures 14 and 15). Hence, there are two orbits, each one of them with two elements. We say that the action of

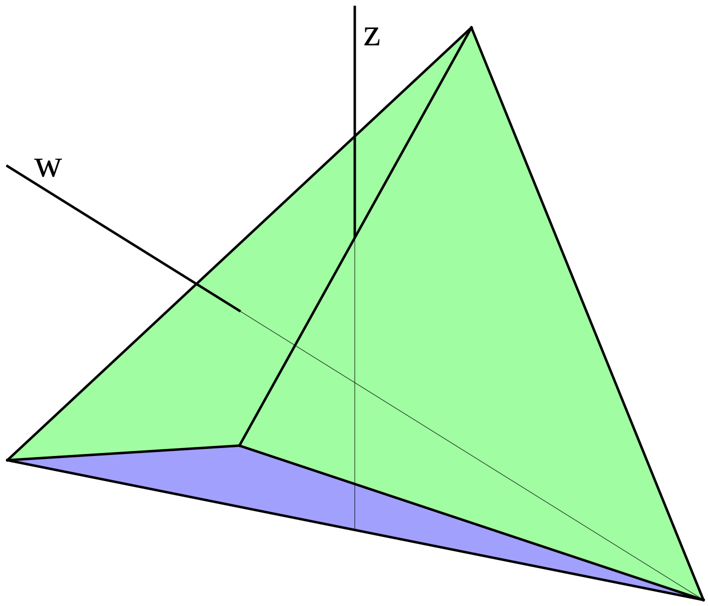

Figure 14.

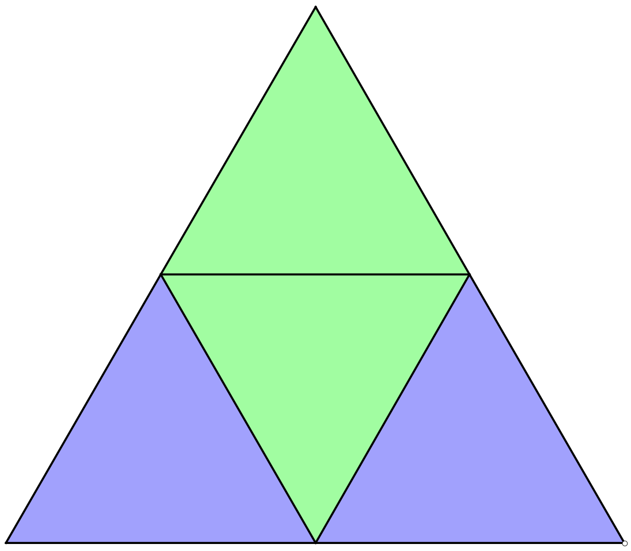

Figure 15.

this subgroup is of the $(1, 2 \times 2)$ type. The first component, the 1, is the determinant of the generator.

b) Consider, in the figures 11 and 16, the rotations of 0°, 120° and 240° around the w-axis. They form a subgroup of Ω^+ of order three. The action of these rotations on the tetrahedron faces keeps the green

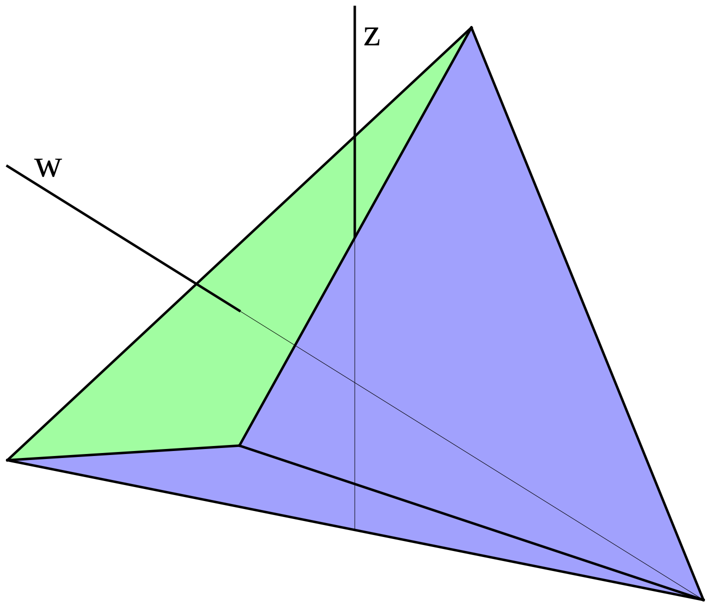

FIGURE 16.

face fixed and sends every blue face, by order of succession, to the place of all other blue faces. The set with the green face and the set of the blue faces are the orbits of this action (see figures 16 and 17). Therefore, there are two orbits, one with three elements , and one with only one element. We say that the action of this subgroup is of the $(1, 1 \times 1 + 1 \times 3)$ type. The first component, the 1, is the determinant of the generator.

3.2. **The octahedron (cube) group.** Consider the octahedron (see figures 18 and 19) and its group, Ω. Everything that we say here about the octahedron group can be translated to the cube group interchanging faces with vertices. In other words the group is the same.

An element of Ω is, for example, the central symmetry $\omega(x, y, z) = -(x, y, z)$, that induces the function $\omega : E \longrightarrow E$

$$\begin{array}{cccc} \omega(e_1) = e_{11} & \omega(e_2) = e_{12} & \omega(e_3) = e_9 & \omega(e_4) = e_{10} \\ \omega(e_5) = e_7 & \omega(e_6) = e_8 & \omega(e_7) = e_5 & \omega(e_8) = e_6 \\ \omega(e_9) = e_3 & \omega(e_{10}) = e_4 & \omega(e_{11}) = e_1 & \omega(e_{12}) = e_2. \end{array}$$

The central symmetry has determinant -1. The symmetries with determinant 1 (Ω^+) can be seen like this: one transports a chosen face in such a way that it goes to one of the eight faces of the octahedron; as one has three possibilities of making them coincide (they are equilateral triangles), there are 24 (3×8) symmetries with determinant 1. Figure

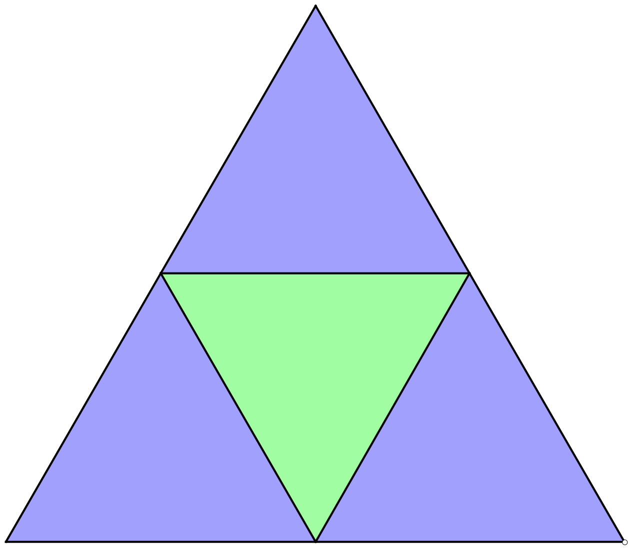

FIGURE 17.

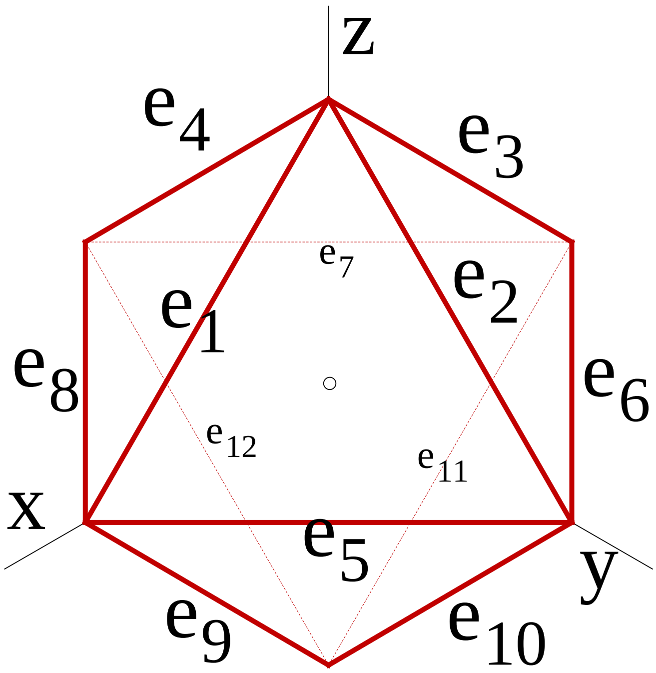

FIGURE 18.

20 shows one of these symmetries. In this case the function $\omega : E \to E$, is the following:

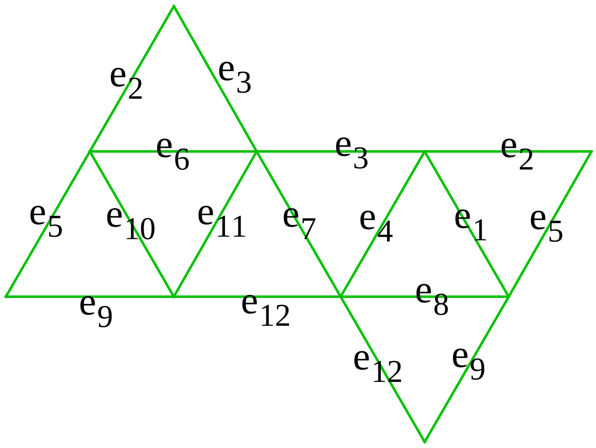

FIGURE 19.

$$\begin{array}{llll}\omega(e_1)=e_4 & \omega(e_2)=e_7 & \omega(e_3)=e_{12} & \omega(e_4)=e_8\\ \omega(e_5)=e_3 & \omega(e_6)=e_{11} & \omega(e_7)=e_9 & \omega(e_8)=e_1\\ \omega(e_9)=e_2 & \omega(e_{10})=e_6 & \omega(e_{11})=e_{10} & \omega(e_{12})=e_5.\end{array}$$

Note once more that the advantage of describing in this way the symmetries of Ω^+ is that it can be easily adapted to other polyhedra, and used in their computation in a computer program.

Another way of counting the symmetries de Ω^+ is the following: the identity (1); the rotations of 90°, 180° and 270° around the three axes defined by opposite vertices (9); the rotations of 180° around the six axes defined by the centers of opposite edges (6); the rotations of 120° and 240° around the four axes defined by the centers of opposite faces (8).

The symmetries with determinant -1 (Ω^-) are the compositions of the symmetries de Ω^+ with the central symmetry. The cardinal of Ω, the order of Ω, is, therefore, 48.

In reference [5] one can see two examples of such symmetries and a detailed description of the octahedron puzzle case.

3.3. **The icosahedron (dodecahedron) group.** Consider the icosahedron (see figure 21) and its group, Ω. Everything that we say here about the icosahedron group can be translated to the dodecahedron group interchanging faces with vertices. In other words the group is the same.

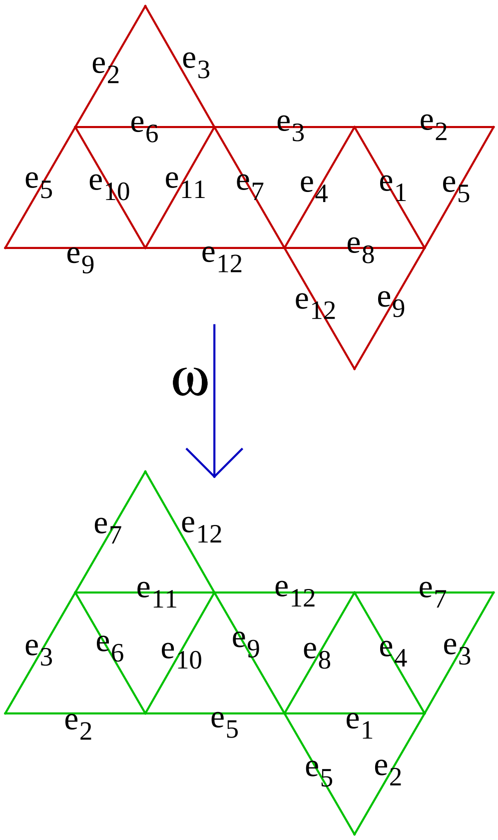

FIGURE 20.

An element of Ω is, for example, the central symmetry $\omega(x, y, z) = -(x, y, z)$, that induces the function $\omega : E \longrightarrow E$

$$\begin{array}{ccccc}
\omega(e_1) = e_{16} & \omega(e_2) = e_{17} & \omega(e_3) = e_{18} & \omega(e_4) = e_{19} & \omega(e_5) = e_{20} \\
\omega(e_6) = e_{21} & \omega(e_7) = e_{22} & \omega(e_8) = e_{23} & \omega(e_9) = e_{24} & \omega(e_{10}) = e_{25} \\
\omega(e_{11}) = e_{26} & \omega(e_{12}) = e_{27} & \omega(e_{13}) = e_{28} & \omega(e_{14}) = e_{29} & \omega(e_{15}) = e_{30} \\
\omega(e_{16}) = e_1 & \omega(e_{17}) = e_2 & \omega(e_{18}) = e_3 & \omega(e_{19}) = e_4 & \omega(e_{20}) = e_5 \\
\omega(e_{21}) = e_6 & \omega(e_{22}) = e_7 & \omega(e_{23}) = e_8 & \omega(e_{24}) = e_9 & \omega(e_{25}) = e_{10} \\
\omega(e_{26}) = e_{11} & \omega(e_{27}) = e_{12} & \omega(e_{28}) = e_{13} & \omega(e_{29}) = e_{14} & \omega(e_{30}) = e_{15}.
\end{array}$$

The central symmetry has determinant -1. The symmetries with determinant 1 (Ω^+) can be seen like this: one transports a chosen face in such a way that it goes to one of the twenty faces of the icosahedron; as one has three possibilities of making them coincide (they are equilateral triangles), there are 60 (3×20) symmetries with determinant 1. Figure

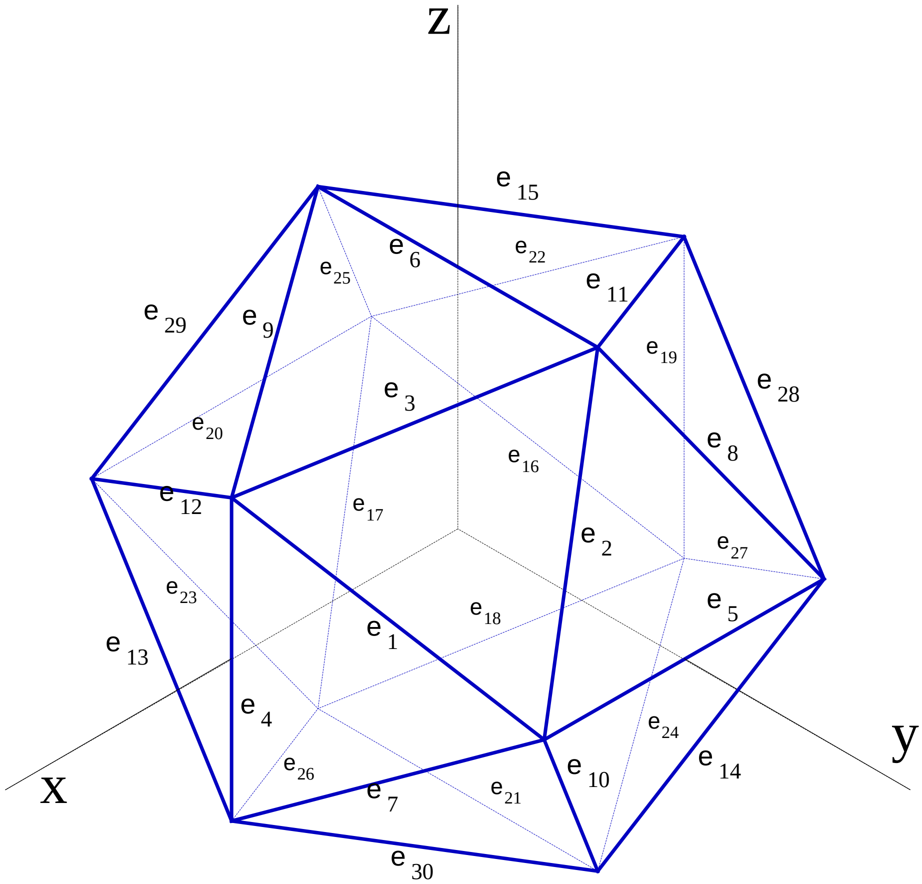

FIGURE 21.

22 shows one of these symmetries. In this case the function $\omega : E \to E$, is the following:

$$
\begin{array}{lllll}
\omega(e_1) = e_5 & \omega(e_2) = e_8 & \omega(e_3) = e_2 & \omega(e_4) = e_{10} & \omega(e_5) = e_{28} \\
\omega(e_6) = e_3 & \omega(e_7) = e_{14} & \omega(e_8) = e_{11} & \omega(e_9) = e_1 & \omega(e_{10}) = e_{27} \\
\omega(e_{11}) = e_6 & \omega(e_{12}) = e_7 & \omega(e_{13}) = e_{30} & \omega(e_{14}) = e_{19} & \omega(e_{15}) = e_9 \\
\omega(e_{16}) = e_{20} & \omega(e_{17}) = e_{23} & \omega(e_{18}) = e_{17} & \omega(e_{19}) = e_{25} & \omega(e_{20}) = e_{13} \\
\omega(e_{21}) = e_{18} & \omega(e_{22}) = e_{29} & \omega(e_{23}) = e_{26} & \omega(e_{24}) = e_{16} & \omega(e_{25}) = e_{12} \\
\omega(e_{26}) = e_{21} & \omega(e_{27}) = e_{22} & \omega(e_{28}) = e_{15} & \omega(e_{29}) = e_4 & \omega(e_{30}) = e_{24}.
\end{array}
$$

Another way of counting the symmetries de Ω^+ is the following: the identity (1); the rotations of 72°, 144° , 216° and 288° around the six axes defined by opposite vertices (24); the rotations of 180° around the fifteen axes defined by the centers of opposite edges (15); the rotations

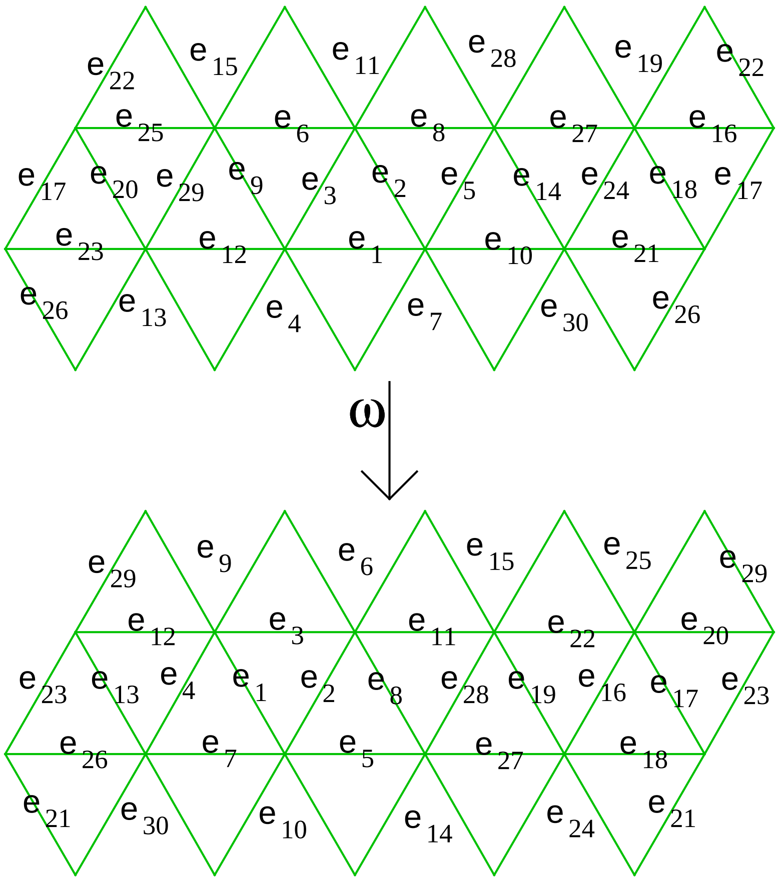

FIGURE 22.

of 120° and 240° around the ten axes defined by the centers of opposite faces (20).

The symmetries with determinant -1 (Ω^-) are the compositions of the symmetries de Ω^+ with the central symmetry. The cardinal of Ω, the order of Ω, is, therefore, 120.

4. Permutation groups and puzzle solutions

Consider a puzzle with numbers $1, 2, \ldots, n$ drawn on the plates. From now on P denotes the set of its plates which have numbers drawn, and call it the plate set. If no confusion is possible, P will also denote the puzzle itself. S_n denotes the group of all permutations of $\{1, 2, \ldots, n\}$; $\sigma \in S_n$ means that σ is a one-to-one function $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$. The identity is σ_0: $\sigma_0(1) = 1, \sigma_0(2) = 2, \ldots, \sigma_0(n) = n$. The alternating group, the S_n subgroup of the even permutations, is denoted by A_n. If $\sigma_1, \sigma_2 \in S_n$, we shall denote $\sigma_1\sigma_2 \equiv \sigma_1 \circ \sigma_2$.

We shall write $\sigma = (\alpha_1\alpha_2\cdots\alpha_k)\cdots(\beta_1\beta_2\cdots\beta_l)$, if

$$\begin{aligned}&\sigma(\alpha_1)=\alpha_2,\ \sigma(\alpha_2)=\alpha_3,\ \ldots,\ \sigma(\alpha_k)=\alpha_1,\\&\ldots,\\&\sigma(\beta_1)=\beta_2,\ \sigma(\beta_2)=\beta_3,\ \ldots,\ \sigma(\beta_k)=\beta_1,\end{aligned}$$

where $\alpha_1,\alpha_2,\ldots,\alpha_k,\ldots,\beta_1,\beta_2,\ldots,\beta_l \in \{1,2,\ldots,n\}$.

If $\gamma \in \{1,2,\ldots,n\}\setminus\{\alpha_1,\alpha_2,\ldots,\alpha_k,\ldots,\beta_1,\beta_2,\ldots,\beta_l\}$, then $\sigma(\gamma)=\gamma$.

The permutation $(\alpha_1\alpha_2\cdots\alpha_k)$ is called a cyclic permutation, or a cycle (in this case a k-cycle); k is the length of the cyclic permutation.

We shall use also the group $\{-1,1\}\times S_n$ denoted by $S_n^{\pm}$. If $\delta_1,\delta_2\in\{-1,1\}$ and $\sigma_1,\sigma_2\in S_n$, then $(\delta_1,\sigma_1)(\delta_2,\sigma_2)=(\delta_1\delta_2,\sigma_1\sigma_2)$. We denote $S_n^+=\{1\}\times S_n\equiv S_n$, $(1,\sigma)\equiv\sigma$, $(-1,\sigma)\equiv\sigma^-$.

As before E denotes the set of the polyhedron edges and F denotes the set of the polyhedron faces. A solution of the puzzle defines a function $\varepsilon : E \to \{1,2,\ldots,n\}$. Denote $\mathcal{E}$ the set of these functions. One can say that $\mathcal{E}$ is the set of the puzzle solutions.

We shall also consider the group $S_n\times\Omega$. If $(\sigma_1,\omega_1),(\sigma_2,\omega_2)\in S_n\times\Omega$, one defines the product $(\sigma_1,\omega_1)(\sigma_2,\omega_2)=(\sigma_1\sigma_2,\omega_1\omega_2)$. We use here a different definition from the one in reference [4].

4.1. **The plate group.** Some S_n subgroups act naturally on P. Let $\pi\in P$ and $\sigma\in S_n$. Assume that $a,b,c,\ldots$ are drawn on π, by this order. Then $\sigma\pi$ is a plate where the numbers $\sigma(a)=a_1,\sigma(b)=b_1,\sigma(c)=c_1,\ldots$ are drawn replacing $a,b,c,\ldots$ (see figure 23).

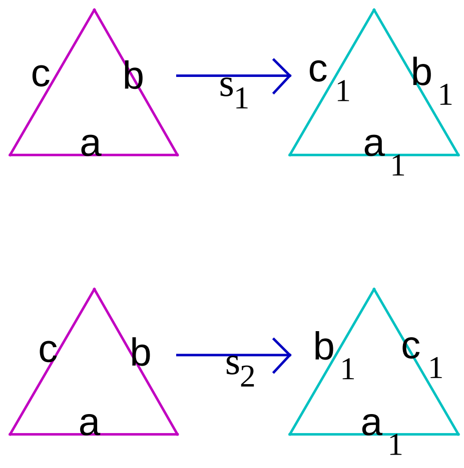

FIGURE 23.

Let $s\in S_n^{\pm}$ and $\pi\in P$. If $s\equiv s_1=(1,\sigma)\equiv\sigma$, then $s\pi=\sigma\pi$. If $s\equiv s_2=(-1,\sigma)\equiv\sigma^-$, then $s\pi$ is a reflection of $\sigma\pi$. In this last case, if the numbers $a, b, c, \ldots$ are drawn on π, by this order, then $s\pi$ is a plate where the numbers $\ldots, \sigma(c) = c_1, \sigma(b) = b_1, \sigma(a) = a_1$ are drawn by this order (see figure 23).

The plate group, G_P, is the greatest subgroup of $S_n^{\pm}$ that acts on P. If $s \in S_n^{\pm}$ and $s\pi \in P$, for every $\pi \in P$, then $s \in G_P$.

4.2. **The solution group.** Let $\varepsilon : E \to \{1, 2, \ldots, n\}$ be a solution of the puzzle. The group of this solution, G_ε, is a subgroup of $S_n \times \Omega$; $(\sigma, \omega) \in G_\varepsilon$ if and only if

$$\sigma \circ \varepsilon = \varepsilon \circ \omega.$$

Denote by Ω_ε the following subgroup of Ω: $\omega \in \Omega_\varepsilon$ if and only if there exists $\sigma \in S_n$ such that $(\sigma, \omega) \in G_\varepsilon$. Notice that if $\omega \in \Omega_\varepsilon$ there exists only one $\sigma \in S_n$ such that $(\sigma, \omega) \in G_\varepsilon$. From this one concludes that $\omega \mapsto (\sigma, \omega)$ defines an isomorphism between Ω_ε and G_ε and that $(\det \omega, \sigma) \in G_P$. This defines $g_\varepsilon : \Omega_\varepsilon \to G_P$, $g_\varepsilon(\omega) = (\det \omega, \sigma)$, which is an homomorphism of groups.

For a lot of puzzles $(\det \omega, \sigma)$ defines completely ω. It is the case of all puzzles considered in this article. Hence, when $(\det \omega, \sigma)$ defines completely ω, g_ε establishes an isomorphism between Ω_ε and $g_\varepsilon(\Omega_\varepsilon) \subset G_P$. Denote by $G_{P\varepsilon} \equiv g_\varepsilon(\Omega_\varepsilon)$. Finally, G_ε and $G_{P\varepsilon}$ are isomorphic. We can identify (σ, ω) with $(\det \omega, \sigma)$, and G_ε with the subgroup $G_{P\varepsilon}$ of G_P.

4.3. **Equivalent solutions.** Let $\varepsilon_1, \varepsilon_2 : E \to \{1, 2, \ldots, n\}$ be solutions of the puzzle. One says that these solutions are equivalent, $\varepsilon_1 \approx \varepsilon_2$, if there are $\omega \in \Omega$ and $\sigma \in S_n$ such that

$$\sigma \circ \varepsilon_1 = \varepsilon_2 \circ \omega.$$

Notice that $(\det \omega, \sigma) \in G_P$.

If $\sigma = \sigma_0$ and $\det \omega = 1$, what distinguishes the solutions ε_1 and ε_2 is only a rotational symmetry. In this case,

$$\varepsilon_1 = \varepsilon_2 \circ \omega$$

expresses another equivalence relation, $\varepsilon_1 \sim \varepsilon_2$. When we make a puzzle, in practice, we do not recognize the difference between ε_1 and ε_2. We shall say that they represent the same **natural solution**, an equivalence class of the relation $\sim$.

Figure 24 shows two solutions of the octahedron puzzle that represent the same natural solution.

Let $\varepsilon, \varepsilon_1, \varepsilon_2 \in \mathcal{E}$. As $\varepsilon_1 \sim \varepsilon_2$ and $\varepsilon_1 \approx \varepsilon$ imply $\varepsilon_2 \approx \varepsilon$, one can say that the natural solution represented by ε_1 is equivalent to ε.

This equation involving ε_1 and ε_2 defines an equivalence relation, and a natural solution is an equivalence class of this relation. Notice that if $\varepsilon_1 = \varepsilon_2$, then ω_1 is the identity.

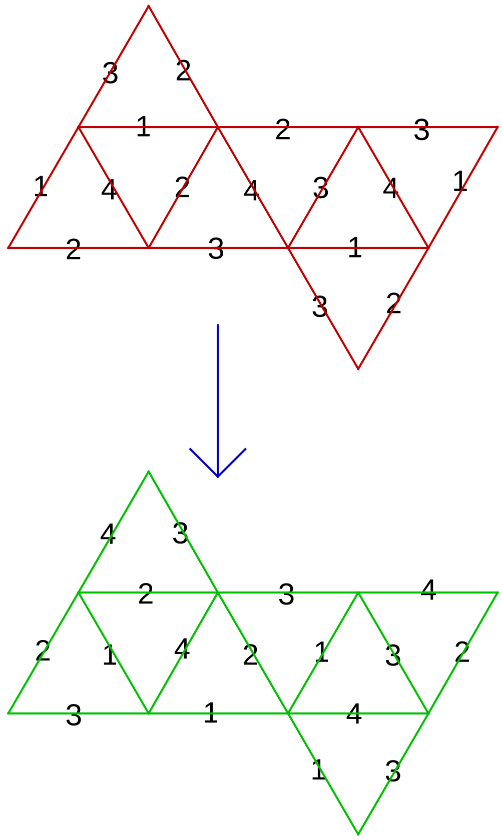

FIGURE 24.

4.4. **Maximal solutions. Maximal puzzles.** If ε_1 and ε_2 are two solutions of a puzzle, we define the relation $\varepsilon_1 \succeq \varepsilon_2$ when $G_{\varepsilon_1} \supset G_{\varepsilon_2}$, or equivalently, when $\Omega_{\varepsilon_1} \supset \Omega_{\varepsilon_2}$. One says that a solution ε of a puzzle is maximal if it is maximal for this partial order relation. Every polyhedron puzzle P has a set of Ω subgroups $\{\Omega_1, \Omega_2, \cdots, \Omega_k\}$ which are the groups of its maximal solutions. We call it the set of maximal groups of P.

If P_1 and P_2 are two puzzles of the same polyhedron, we define the relation $P_1 \succeq P_2$ when every maximal group of P_2 is contained in a maximal group of P_1. One says that a puzzle P is maximal if it is maximal for this partial order relation.

4.5. **Equivalent puzzles.** Consider two puzzles and their plate sets, P_1 and P_2. One says that they are equivalent if there exists $s \in S_n^{\pm}$ such that the function $\pi \mapsto s\pi$ is one-to-one between P_1 and P_2. We denote P_2 as sP_1.

As a first example take the dodecahedron (2) puzzle and its plate set P (figure 9). If $s = (1, \sigma)$ where σ is a transposition, then sP is the plate set of an equivalent puzzle (figure 10). Note that $P \neq sP$. If ε is a solution of the dodecahedron (2) puzzle, then $\sigma \circ \varepsilon$ is a solution of the other puzzle.

Another example concerns the icosahedron (3) puzzle and its plate set P (figure 4). If $s = (-1, \sigma_0)$, where σ_0 is the identity, then sP is the plate set of an equivalent puzzle (figure 5). As before $P \neq sP$ and if ε is a solution of the icosahedron (3) puzzle, then $\varepsilon \circ \omega$, where ω is the central symmetry, is a solution of the other puzzle.

In practice, equivalent puzzles are viewed as the same puzzle.

4.6. **Duality. Dual puzzles.** If one has a solution of any puzzle P over a polyhedron, one can construct the dual of the solution over the polyhedron dual and, therefore, have another puzzle P'. Equivalent solutions ε_1 and ε_2 of the puzzle P, correspond to equivalent puzzles P'_1 and P'_2, as if $\sigma \circ \varepsilon_1 = \varepsilon_2 \circ \omega$, then $P'_2 = (\det \omega, \sigma) P'_1$.

Consider a puzzle P over a polyhedron and its maximal solutions $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$ and construct the puzzles $P' \equiv P'_1, P'_2, \ldots, P'_k$, over the polyhedron dual, that correspond to that solutions. Assume that these puzzles are equivalent. Use the same procedure with P' in order to produce $P''_1, P''_2, \ldots, P''_j$ and assume that these and P are equivalent puzzles. Then we say that P and P' are dual puzzles.

The icosahedron (1) and the dodecahedron (2) puzzles are dual. The cuboctahedron (1) puzzle and the rhombic dodecahedron puzzle of reference [6] are also dual.

5. From groups to polyhedron puzzles

Let S be a subgroup of $S_n^{\pm}$ and $h_1, h_2 : \Omega \to S$ be two isomorphisms.

One says that h_1 and h_2 represent the same natural isomorphism if there exists $\omega \in \Omega^+$ such that

$$h_1 = h_2 \circ \omega.$$

Two natural isomorphisms, represented by h_1 and h_2 are said to be equivalent if there exists $s \in S$ and $\omega \in \Omega$, such that

$$s \circ h_1 = h_2 \circ \omega.$$

Consider a puzzle solution ε and its group $G_{P_\varepsilon} \equiv \Omega_\varepsilon$. Take $\omega_1, \omega_2 \in \Omega_\varepsilon$. If ω_1 is a rotation of order k, and ω_2 is a rotation of order j, then ω_1 transforms ω_2 in another rotation of order j, ω_3, which is $\omega_1 \omega_2 \omega_1^{-1} \equiv \omega_1(\omega_2)$.

The isomorphism between Ω_ε and G_{P_ε} suggests that if one wants to translate the isometries into elements of $S_n^{\pm}$ the function must be such that if $\omega_1 \mapsto s_1$, $\omega_2 \mapsto s_2$, then $\omega_3 \equiv \omega_1(\omega_2) \mapsto s_1 s_2 s_1^{-1}$.

Note that if $s_1 = (\delta_1, \sigma_1)$, $s_2 = (\delta_2, \sigma_2)$, where

$$\sigma_2 = (\alpha_1 \alpha_2 \cdots \alpha_l) \cdots,$$

then $\omega_3 \equiv \omega_1(\omega_2) \longmapsto s_3 = (\delta_2, \sigma_3)$, with

$$\sigma_3 = (\sigma_1(\alpha_1)\, \sigma_1(\alpha_2) \cdots \sigma_1(\alpha_l)) \cdots.$$

We assign to every semiaxis of order k ($\equiv \omega$) a k-cycle $\sigma = (\alpha_1 \alpha_2 \cdots \alpha_k)$, so that to the counter clock-wise rotation of $\frac{2\pi}{k}$, ω, corresponds the permutation σ. This association must be coherent in the sense that it generates a group isomorphism.

In the cases we are interested in (the tetrahedron, the octahedron and the icosahedron), it is enough to make the association to two neighbor semiaxes. In these cases the semiaxes are defined by the vertices, the edges (the middle point of each edge), the faces (the center of each face), and have their origin at the polyhedron center.

If we assign to ω and ω_1 (two neighbor semiaxes) the cycles σ and σ_1, then to the semiaxis $\omega(\omega_1) = \omega\omega_1\omega^{-1}$ we must assign $\sigma\sigma_1\sigma^{-1}$. When $\sigma_1 = (\alpha_1 \alpha_2 \cdots \alpha_k)$, then $\sigma\sigma_1\sigma^{-1} = (\sigma(\alpha_1)\sigma(\alpha_2) \cdots \sigma(\alpha_k))$.

In this section we use group theory in order to find puzzles, for a given polyhedron, such as, for example, maximal puzzles. These are important examples, but others could be given.

To avoid ambiguities, in the puzzles we give in the following, all the edges have numbers, and we use the numbers $1, 2, 3, 4$ in the tetrahedron and octahedron (cube) groups cases, and the numbers $1, 2, 3, 4, 5$ in the icosahedron (dodecahedron) group case.

5.1. **The tetrahedron group.** In the tetrahedron a vertex represents a semiaxis of order 3. Associate to a given vertex the permutation (123). It is not difficult to see that the only natural possibility is the one represented in the l.h.s. in figure 25. Something similar happens with permutation (132) and the result is in the r.h.s. of the same figure. Hence, there are two natural isomorphisms for the tetrahedron group (see figure 25) which are equivalent. This equivalence can be done, for example, by any transposition and an element of Ω^+. In this case

$$S = (\{1\} \times A_4) \cup (\{-1\} \times (S_4 \setminus A_4)).$$

5.1.1. *Cube puzzles.* Figure 26 shows the two solutions of the cube (2) puzzle, which represent even better than figure 25 the tetrahedron group.

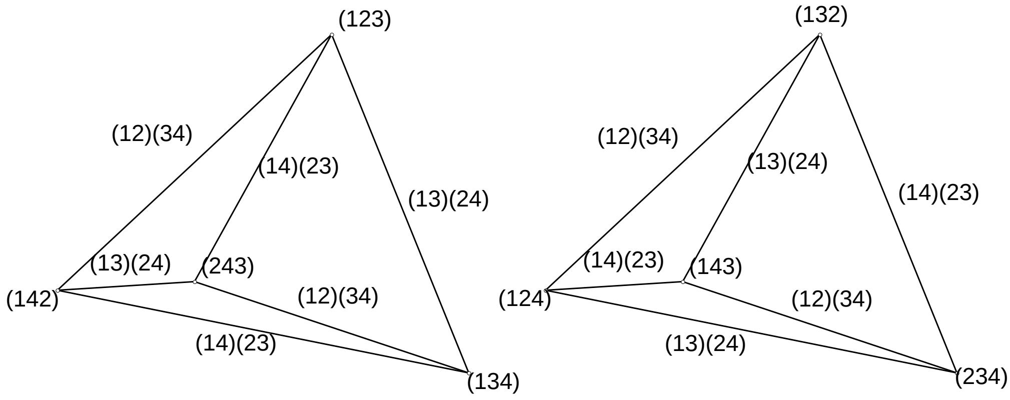

FIGURE 25.

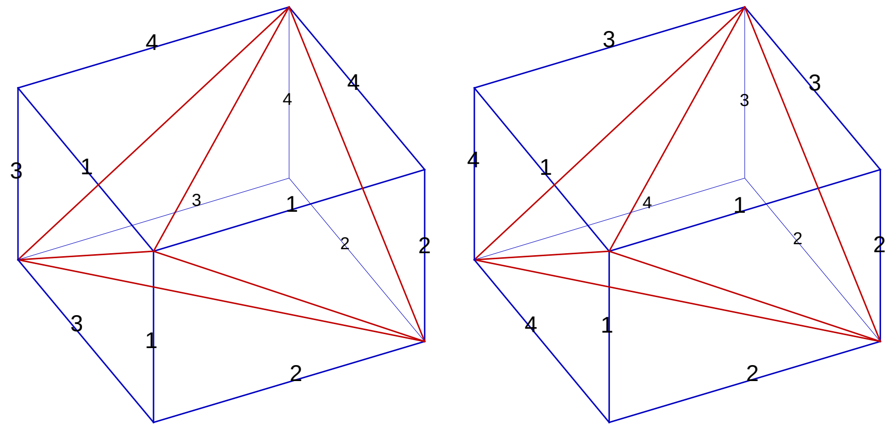

FIGURE 26.

5.1.2. *Octahedron puzzles.* Figure 27 represents solutions of two equivalent puzzles, that we call the octahedron (2) puzzle. These solutions correspond to the natural tetrahedron isomorphisms. Note that

$$S = (\{1\} \times A_4) \cup (\{-1\} \times (S_4 \setminus A_4))$$

is precisely the plate group.

5.1.3. *Cuboctahedron puzzles.* Figure 28 represents solutions of two equivalent puzzles, that we call the cuboctahedron (3) puzzle. These solutions correspond to the natural tetrahedron isomorphisms. Note that, as in the octahedron case, S is precisely the plate group.

5.2. **The octahedron (cube) group.** There is only one natural isomorphism for the octahedron (cube) group (see figure 29). In this case

$$S = \{-1, 1\} \times S_4.$$

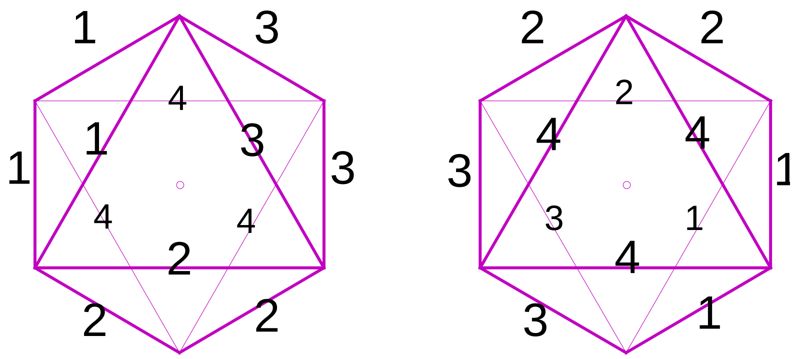

FIGURE 27.

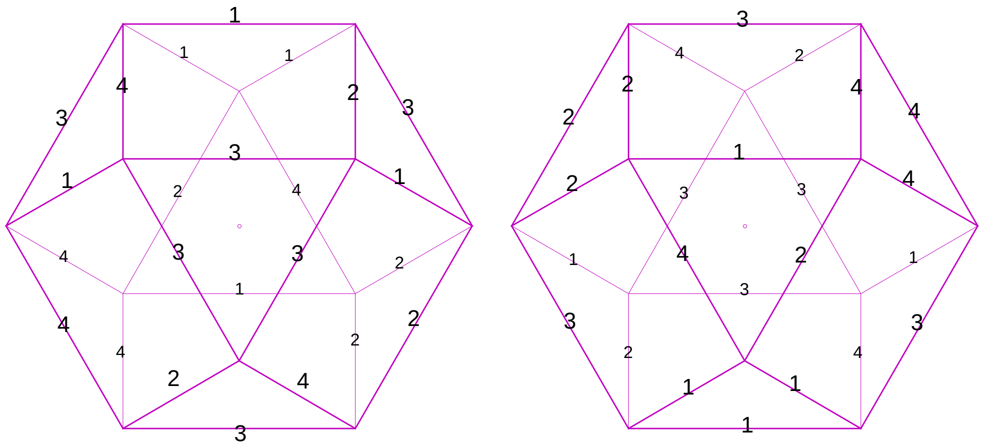

FIGURE 28.

5.2.1. *Cube and octahedron puzzles.* If one looks for puzzles with solutions that have as their group $\Omega^+ \equiv S_4$, we find the cube (1) and the octahedron (1) puzzles. Those solutions are maximal. This means that there are no puzzles where $S = \{-1, 1\} \times S_4$ is a solution group.

5.2.2. *Cuboctahedron puzzles.* The cuboctahedron (1) puzzle has only a maximal natural solution which is shown in the l.h.s. of figure 30. As its group is precisely $S = \{-1, 1\} \times S_4$, this solution is excellent in order to represent the octahedron (cube) group. This puzzle and this solution is completely rediscovered using S.

In the r.h.s. of figure 30 is represented the solution of the cuboctahedron (2) puzzle which has also S as group. These two are the only possibilities for cuboctahedron puzzles with S as maximal group.

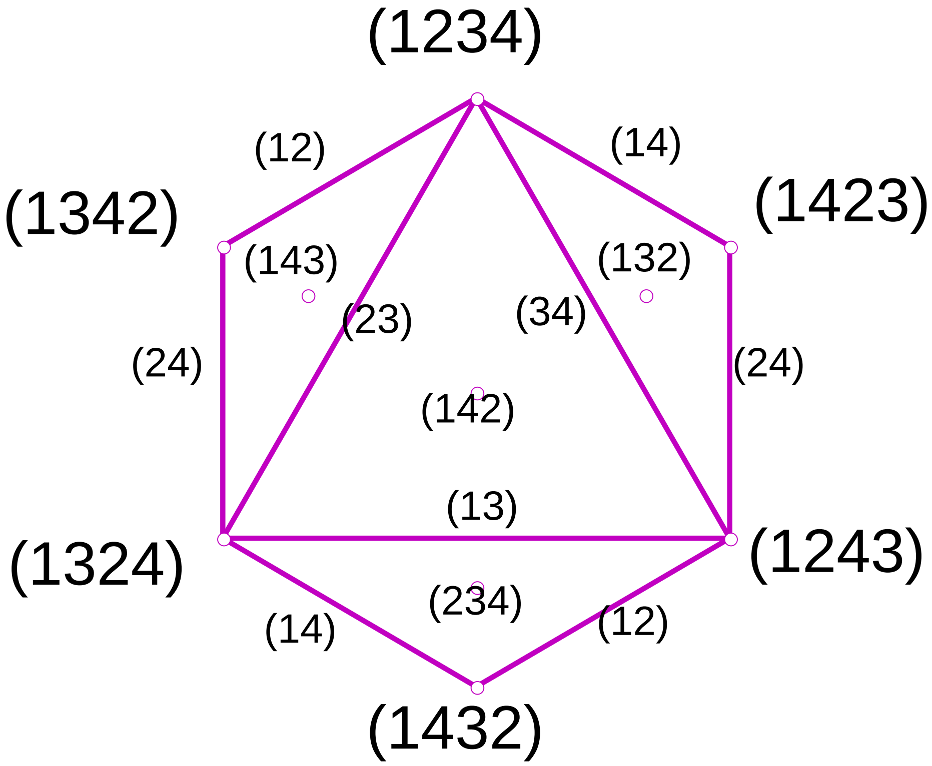

FIGURE 29.

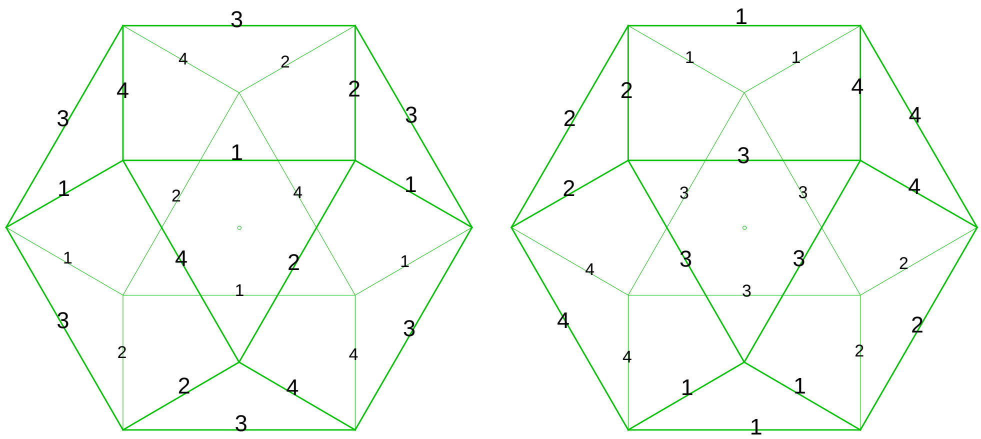

FIGURE 30.

Note that the rhombic dodecahedron puzzle of Reference [6] has only one natural solution which is dual of the l.h.s. solution in figure 30. This property was crucial in choosing the puzzle.

5.2.3. *Rhombicuboctahedron puzzles.* Consider a rhombicuboctahedron square face which has no common edge with a triangular face. Associate to this face the permutation (1234). To a neighbor triangular face one must assign the permutation (132), as it is shown in figure 31, if one wants to have a puzzle with a maximal group containing S_4. Figure 31 shows sixteen possibilities and none of them has a central symmetry. All of them are maximal solutions of puzzles with S_4 as a group.

There are six puzzles. All the puzzles have the six plates of figure 6. Four of them have the eight plates represented in figure 2 (see the first three rows in figure 31) and the other two have eight triangular plates of the type *aaa* (see the last row in figure 31). The first puzzle has the remaining square plates of the type *abac* (6 solutions). The second puzzle has the remaining square plates of the type *aabb* (2 solutions). The third puzzle has the remaining square plates of the type *aaaa* (2 solutions). The forth puzzle has the remaining square plates of the type *abab* (2 solutions). The fifth puzzle has the remaining square plates of the type *abac* (2 solutions). The sixth puzzle has the remaining square plates of the type *aabb* (2 solutions).

The first puzzle is the one presented in Reference [6]. It was chosen because two of its six solutions with S_4 as a maximal group are dual of two maximal solutions of the deltoidal icositetrahedron puzzle presented in the same Reference [6].

5.2.4. *Snub cube puzzles.* As in the rhombicuboctahedron situation there are sixteen cases, that are shown in figure 32, if one wants to have puzzles with S_4 as a maximal group (the largest one possible, as the snub cube has no central symmetry).

In the figure the edge marked with x must have a 2 or a 3 in each of the sixteen cases. Hence, there are, in fact, thirty two possibilities.

There are five puzzles. All the puzzles have the six plates of figure 6. Three have the eight plates represented in figure 2 and two have eight triangular plates of the type *aaa*.

The first puzzle has the remaining triangular plates of the type *abc* (8 solutions). The second puzzle has the remaining triangular plates of the type *aab* (14 solutions). The third puzzle has the remaining triangular plates of the type *aaa* (2 solutions). These are the first three puzzles (the first three columns in figure 32).

The last two puzzles (the last column in figure 32) are as follows. The fourth puzzle has the remaining triangular plates of the type *abc* (4 solutions). The fifth puzzle has the remaining triangular plates of the type *aab* (4 solutions).

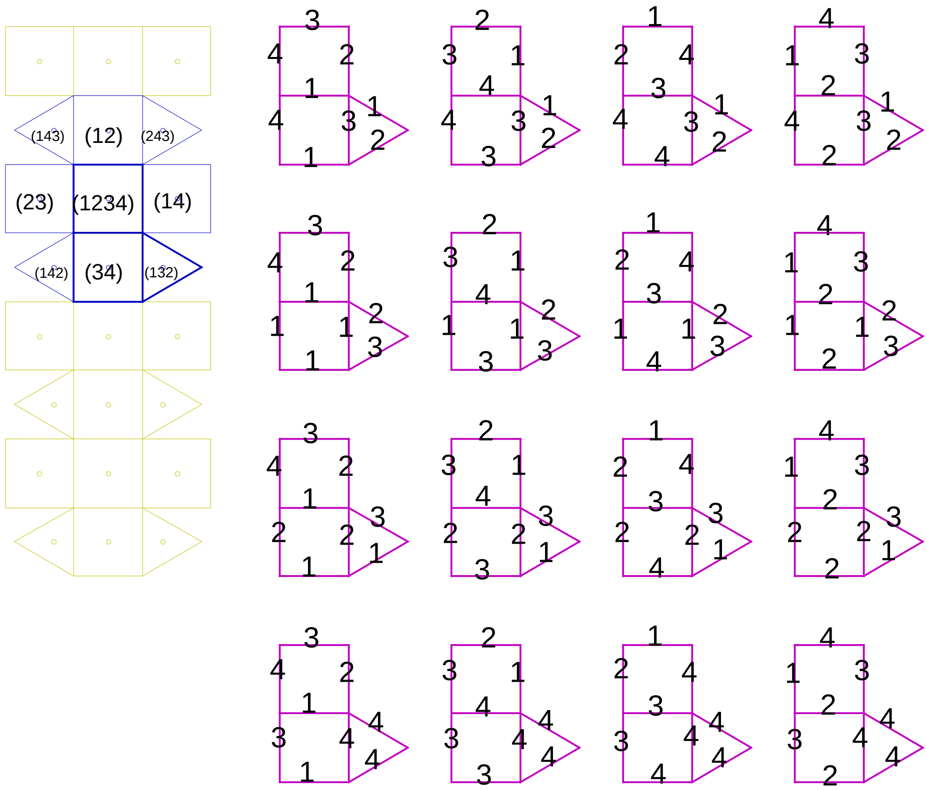

FIGURE 31.

5.3. **The icosahedron (dodecahedron) group.** In the icosahedron the center of a face represents a semiaxis of order 3. Associate to a given face the permutation (123) as it is shown in figure 33. Then there are 8 possibilities for the vertices of this face, but only two of them, the green ones in the figure, are coherent in the sense that they generate a group isomorphism.

There are two natural isomorphisms for the icosahedron (dodecahedron) group (see figure 34) which are equivalent. This equivalence is made, for example, by any transposition and an element of Ω^+. In this case

$$S = \{-1, 1\} \times A_5.$$

5.3.1. *Icosahedron and dodecahedron puzzles.* Using this isomorphism one can recover the icosahedron (1) and the dodecahedron (2) puzzles, which have precisely S as maximal group (see Reference [4]).

5.3.2. *Icosidodecahedron puzzles.* For the icosidodecahedron there is only one puzzle with S as maximal group (in fact, there are two equivalent

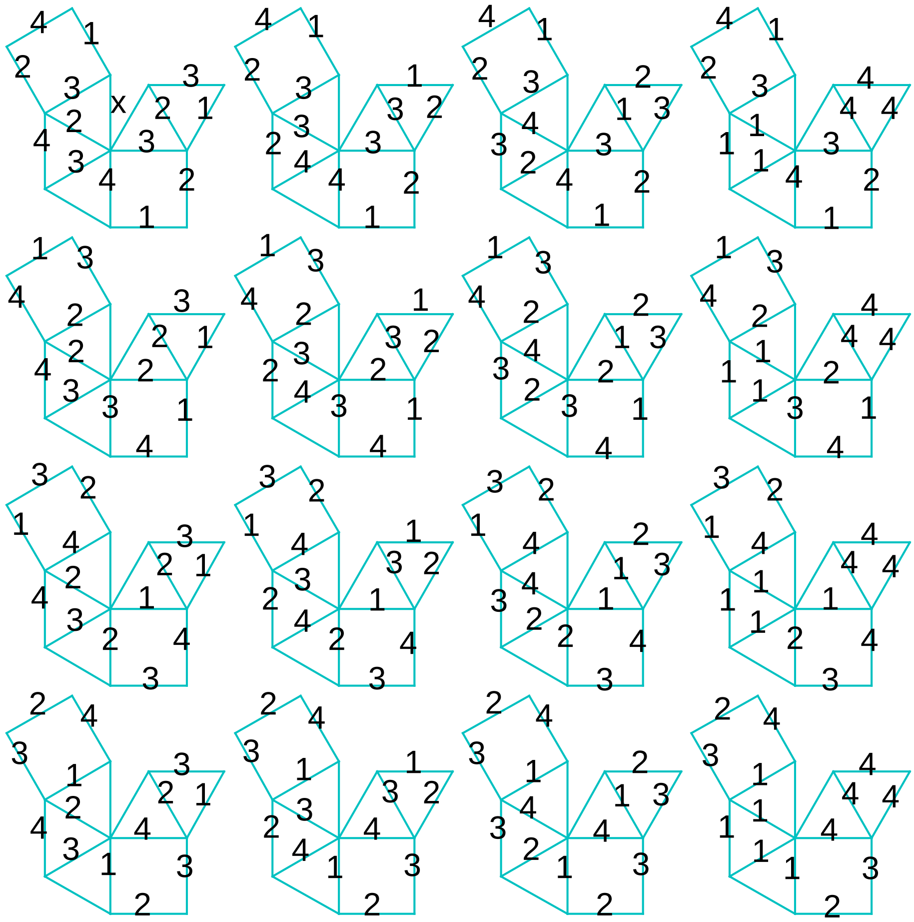

FIGURE 32.

puzzles). Figure 35 shows the possibilities if one wants to have A_5 as a group (two puzzles). Only one of them, the green one, has a central symmetry.

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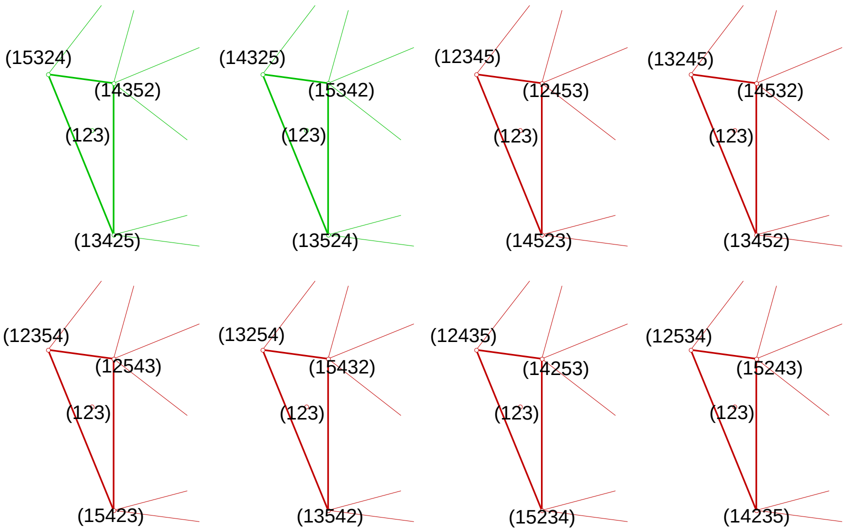

FIGURE 33.

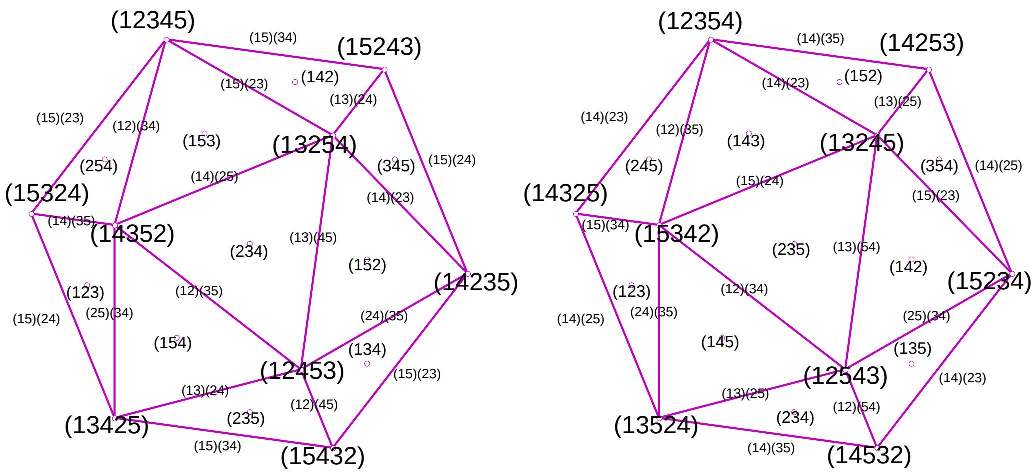

FIGURE 34.

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http://gfm.cii.fc.ul.pt/Members/jr_poliedros-puzzles_en.pdf
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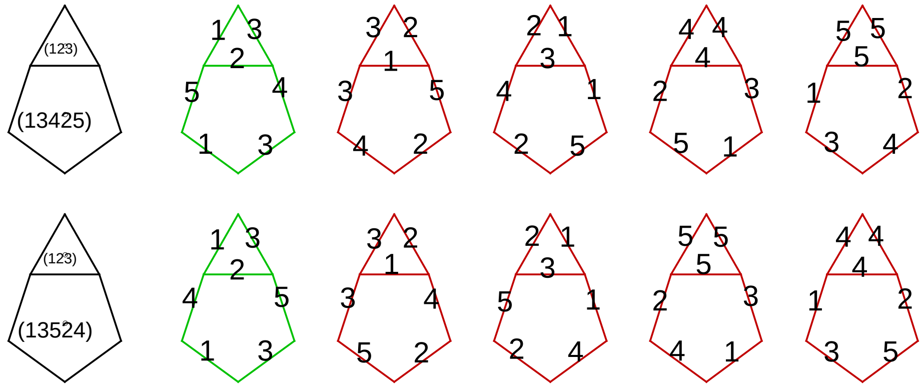

FIGURE 35.

GRUPO DE FÍSICA-MATEMÁTICA DA UNIVERSIDADE DE LISBOA, AV. PROF. GAMA PINTO 2, P-1649-003 LISBOA, PORTUGAL, E DEPARTAMENTO DE MATEMÁTICA, FACULDADE DE CIÊNCIAS DA UNIVERSIDADE DE LISBOA

E-mail address: `rezende@cii.fc.ul.pt`

URL: http://gfm.cii.fc.ul.pt/Members/JR.pt_PT.html